KEY

TO

GREENLEAF'S NEW ELEMENTARY ALGEBRA.

REVISED EDITION.

BY

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KEY

TO

GREENLEAF'S NEW ELEMENTARY

ALGEBRA.

DEFINITIONS AND NOTATION.

SIGNS.

EXAMPLES, pp. 14, 15.

4. Ans. $x + 2y - z$.

8. Ans. $\frac{4a}{3c}$.

9. Ans. $\frac{a - b}{ab}$.

11. Ans. $2a + \frac{b}{c}$.

13. Ans. $\frac{a^4 - b^5}{a - b^2}$.

15. Ans. $\frac{15a^3 + b^5}{a^2 - b^2} + 2c$.

17. Ans. $\frac{1}{ab^2} - \frac{1}{a^2 + c^2}$.

21. Ans. $\sqrt[3]{x} - \sqrt{x}$.

INTERPRETATION OF ALGEBRAIC EXPRESSIONS

EXAMPLES, pp. 16, 17.

1. $12 + 3 - 2 + 4 = 17.$

2. $36 + 2 - 4 = 34.$

3. $48 - 15 + 8 - 28 = 13.$ **Ans. 13.**

4. $(12 - 3) \times (2 + 4) = 9 \times 6 = 54.$

5. $81 \times 2 + 4 = 162 + 4 = 166.$

6. $\frac{15}{5} + 45 = 3 + 45 = 48.$

7. $4 \times 15 - \frac{12}{4} = 60 - 3 = 57.$ **Ans. 57.**

8. $4 \times 9 - 5 \times 4 = 36 - 20 = 16.$

9. $2 \times 144 \times 2 - \frac{144}{2} + \frac{12}{4} = 576 - 72 + 3 = 507.$

10. $\frac{12 + 9 + 8}{13} \times \frac{144 - 27}{9} = \frac{29}{13} \times 13 = 29.$

11. $\left(\frac{11}{11} + 12\right) \times (3 - 2) - 4 = 13 \times 1 - 4 = 9.$

12. $256 - 256 + 12 - 6 = 6.$ **Ans. 6**

13. $252 + 1 \times 1 = 253.$

14. $60 - 7 \times 4 = 60 - 28 = 32.$

15. $4 \times 4 + 8 \times 12 - 112 = 16 + 96 - 112 = 0.$
Ans. 0.

16. $\sqrt{16} + \sqrt{100} - \sqrt{81} = 4 + 10 - 9 = 5.$

17. $12\sqrt{25 - 24} + \sqrt{48 + 16} = 12 + 8 = 20.$

18. $\sqrt{81} - \sqrt[3]{8} + \sqrt{4} = 9 - 2 + 2 = 9.$

19. $10 + 8\sqrt{16} - 2\sqrt[3]{8} = 10 + 32 - 4 = 38.$

ALGEBRAIC PROCESSES.

(Art. **46**, pp. 19, 20.)

2 Let $x =$ number of cents spent,
and $2x =$ number of cents left.
Then $3x = 45$ cents, the whole number,
and $x = 15$ cents, the number spent.

3. Let $x =$ number of apples each has.
Then $2x = 56$ apples, the whole number,
and $x = 28$ apples, the number each has.
Ans. 28 apples.

4. Let $x =$ length of part left standing,
and $3x =$ length of part broken off.
Then $4x = 60$ feet, the whole length,
$x = 15$ feet, the part left standing,
and $3x = 45$ feet, the part broken off.

5. Let $x =$ the less number,
and $5x =$ the greater number.
Then $6x = 126$, their sum,
$x = 21$, the less number,
and $5x = 105$, the greater number.

6. Let $x =$ value of the chaise,
and $3x =$ value of the horse.
Then $4x = 340$ dollars, the value of both,
$x = 85$ dollars, the value of the chaise,
and $3x = 255$ dollars, the value of the horse.

7. Let $x =$ A's share,
and $4x =$ B's share.
Then $5x = 2500$ dollars, the whole sum,
$x = 500$ dollars, A's share,
and $4x = 2000$ dollars, B's share.

(9.)

Let $\quad x =$ A's share,

$2x =$ B's share,

and $\quad 2x =$ C's share.

Then $\quad 5x = 300$ dollars, the whole sum,

$x = 60$ dollars, A's share,

$2x = 120$ dollars, B's share,

and $\quad 2x = 120$ dollars, C's share.

(10.)

Let $\quad x =$ price of the apples,

$2x =$ price of the pears,

and $\quad 4x =$ price of the oranges.

Then $\quad 7x = 63$ cents, the sum paid for the whole,

$x = 9$ cents, the price of the apples,

$2x = 18$ cents, the price of the pears,

and $\quad 4x = 36$ cents, the price of the oranges.

Ans. Apples, 9 cts.; pears, 18 cts.; and oranges, 36 cts

(11.)

Let $\quad x =$ A's age,

$2x =$ B's age,

and $\quad 3x =$ C's age.

Then $\quad 6x = 78$ years, the sum of their ages,

$x = 13$ years, A's age,

$2x = 26$ years, B's age,

and $\quad 3x = 39$ years, C's age.

ADDITION.

(ART. 49, pp. 22, 23.)

(7.)

$$\begin{array}{r} -\ 4\,b\,x \\ -\ 7\,b\,x \\ -\ 3\,b\,x \\ -\ 2\,b\,x \\ -\ 5\,b\,x \\ \hline -21\,b\,x \end{array}$$

(8.)

$$\begin{array}{r} 6\,m\,n^2 \\ 5\,m\,n^2 \\ 1\,m\,n^2 \\ 3\,m\,n^2 \\ 2\,m\,n^2 \\ \hline 17\,m\,n^2 \end{array}$$

(9.)

$$\begin{array}{rl} 2\,a + & b \\ a + & b \\ 4\,a + & b \\ 7\,a + & b \\ 3\,a + & b \\ \hline 17\,a + & 5\,b \end{array}$$

(10.)

$$\begin{array}{rr} 3\,c^2\,d - & a^3\,c \\ c^2\,d - & a^3\,c \\ 2\,c^2\,d - & a^3\,c \\ 5\,c^2\,d - & a^3\,c \\ c^2\,d - & a^3\,c \\ \hline 12\,c^2\,d - & 5\,a^3\,c \end{array}$$

(13.)

$$\begin{array}{rr} 2\,x + & 3\,y \\ x + & 8\,y \\ 3\,x + & y \\ 6\,x + & 2\,y \\ x + & 4\,y \\ 4\,x + & y \\ \hline 17\,x + & 19\,y \end{array}$$

(14.)

$$\begin{array}{rr} 7\,a^2 - & b \\ 3\,a^2 - & 3\,b \\ 6\,a^2 - & 2\,b \\ 2\,a^2 - & b \\ 4\,a^2 - & 6\,b \\ a^2 - & 4\,b \\ \hline 23\,a^2 - & 17\,b \end{array}$$

(ART. 50, pp. 24, 25.)

4. Ans. $24\,a\,x$.

6. Ans. $x^2 - 5\,x\,y^3$.

(7.)

$$\begin{array}{r}
a - b \\
-2\,a - 7\,b \\
7\,a - 2\,b \\
-3\,a + 3\,b \\
-8\,a + b \\
a + 7\,b \\
\hline
-4\,a + b
\end{array}$$

(8.)

$$\begin{array}{r}
5\,c\,d^2 + 7\,a^2\,b \\
-3\,c\,d^2 - 5\,a^2\,b \\
9\,c\,d^2 - 10\,a^2\,b \\
-4\,c\,d^2 + a^2\,b \\
\hline
7\,c\,d^2 - 7\,a^2\,b
\end{array}$$

(ART. 51, page 26.)

(5.)

$$\begin{array}{rrr}
-a\,b^2 & -\,c\,d^2 & \\
-a\,b^2 & +\,c\,d^2 & \\
 & +\,c\,d^2 & -3\,a\,b \\
5\,a\,b^2 & +\,c\,d^2 & \\
\hline
\text{Ans. } 3\,a\,b^2 & +\,2\,c\,d^2 & -3\,a\,b
\end{array}$$

(6.)

$$\begin{array}{rrrr}
3\,x & -7\,y & +2\,z & \\
-x & +4\,y & +6\,z & \\
 & -2\,y & -3\,z & +a \\
4\,x & -y & +3\,z & \\
\hline
6\,x & -6\,y & +8\,z & +a
\end{array}$$

(7.)

$$\begin{array}{rrrr}
-5\,ax & +2\,by & -7 & \\
 & 3\,by & +18 & -4\,z \\
4\,ax & -by & -9 & \\
3\,ax & -2\,by & +26 & \\
\hline
2\,ax & +2\,by & +28 & -4\,z
\end{array}$$

(8.)

$$\begin{array}{rrrrr}
x^3 & +\,ax^2 & +\,bx & +2 & \\
3\,x^3 & -4\,ax^2 & & +7 & -6\,bxy \\
3\,x^3 & -3\,ax^2 & -7\,bx & -19 & \\
\hline
7\,x^3 & -6\,ax^2 & -6\,bx & -10 & -6\,bxy
\end{array}$$

(9.)

$$\begin{array}{rrrr}
8\,a^2\,x^2 & -3\,a\,x & & \\
 & 7\,a\,x & -5\,x\,y & \\
 & -5\,a\,x & +9\,x\,y & -b^2\,c^3 \\
2\,a^2\,x^2 & & +x\,y & \\
\hline
10\,a^2\,x^2 & -a\,x & +5\,x\,y & -b^2\,c^3
\end{array}$$

(Art. 52, page 27.)

(2.)

$$\begin{array}{r} 5\,(a+x) \\ 6\,(a+x) \\ 8\,(a+x) \\ 3\,(a+x) \\ (a+x) \\ \hline 23\,(a+x) \end{array}$$

(3.)

$$\begin{array}{r} 3\,(x^2-a) \\ 2\,(x^2-a) \\ -\,(x^2-a) \\ 6\,(x^2-a) \\ (x^2-a) \\ \hline \text{Ans. } 11\,(x^2-a) \end{array}$$

(4.)

$$\begin{array}{r} 4\sqrt{a-x} \\ 3\sqrt{a-x} \\ -7\sqrt{a-x} \\ 2\sqrt{a-x} \\ \sqrt{a-x} \\ \hline 3\sqrt{a-x} \end{array}$$

(5.)

$$\begin{array}{r} 7\,y-4\,(a+b) \\ 6\,y+2\,(a+b) \\ 2\,y+\,(a+b) \\ y-3\,(a+b) \\ \hline 16\,y-4\,(a+b) \end{array}$$

(6.)

$$\begin{array}{l} 2\,(x-y)^2 \\ 3\,(x-y)^2 \\ \,(x-y)^2 \\ -\,(x-y)^2+\,(x+y) \\ (x+y) \\ \hline 5\,(x-y)^2+2\,(x+y) \end{array}$$

(Art. 53, page 28.)

(2.)

$$\begin{array}{r} b\,x \\ a\,b\,x \\ 2\,c\,x \\ \hline (b+a\,b+2\,c)\,x \end{array}$$

(3.)

$$\begin{array}{r} 3\,a\,y \\ -\,c\,y \\ -\,2\,a\,y \\ \hline (a-c)\,y \end{array}$$

(4.)

$$\begin{array}{r} (a+b)\,x \\ (a-c)\,x \\ \hline (2\,a+b-c)\,x \end{array}$$

(5.)

$$\begin{array}{r} (a+b)\,x \\ 2\,c\,x \\ 2\,x \\ \hline (a+b+2\,c+2)\,x \end{array}$$

(6.)

$$\begin{array}{l} a\,x+b \\ c\,x+d \\ \hline (a+c)\,x+b+d \end{array}$$

(ART. 50, pp. 24, 25.)

4. Ans. $24\,a\,x$.

6. Ans. $x^2 - 5\,x\,y^3$.

(7.)

$$\begin{array}{r}
a - b \\
-2a - 7b \\
7a - 2b \\
-3a + 3b \\
-8a + b \\
a + 7b \\
\hline
-4a + b
\end{array}$$

(8.)

$$\begin{array}{r}
5cd^2 + 7a^2b \\
-3cd^2 - 5a^2b \\
9cd^2 - 10a^2b \\
-4cd^2 + a^2b \\
\hline
7cd^2 - 7a^2b
\end{array}$$

(ART. 51, page 26.)

(5.)

$$\begin{array}{rrr}
-ab^2 & -cd^2 & \\
-ab^2 & +cd^2 & \\
 & +cd^2 & -3ab \\
5ab^2 & +cd^2 & \\
\hline
\text{Ans. } 3ab^2 & +2cd^2 & -3ab
\end{array}$$

(6.)

$$\begin{array}{rrrr}
3x & -7y & +2z & \\
-x & +4y & +6z & \\
 & -2y & -3z & +a \\
4x & -y & +3z & \\
\hline
6x & -6y & +8z & +a
\end{array}$$

(7.)

$$\begin{array}{rrrr}
-5ax & +2by & -7 & \\
 & 3by & +18 & -4z \\
4ax & -by & -9 & \\
3ax & -2by & +26 & \\
\hline
2ax & +2by & +28 & -4z
\end{array}$$

(8.)

$$\begin{array}{rrrrr}
x^3 & +ax^2 & +bx & +2 & \\
3x^3 & -4ax^2 & & +7 & -6bxy \\
3x^3 & -3ax^2 & -7bx & -19 & \\
\hline
7x^3 & -6ax^2 & -6bx & -10 & -6bxy
\end{array}$$

(9.)

$$\begin{array}{rrrr}
8a^2x^2 & -3ax & & \\
 & 7ax & -5xy & \\
 & -5ax & +9xy & -b^2c^3 \\
2a^2x^2 & & +xy & \\
\hline
10a^2x^2 & -ax & +5xy & -b^2c^3
\end{array}$$

(ART. 52, page 27.)

(2.)

$$\begin{array}{r} 5\,(a+x) \\ 6\,(a+x) \\ 8\,(a+x) \\ 3\,(a+x) \\ (a+x) \\ \hline 23\,(a+x) \end{array}$$

(3.)

$$\begin{array}{r} 3\,(x^2-a) \\ 2\,(x^2-a) \\ -\,(x^2-a) \\ 6\,(x^2-a) \\ (x^2-a) \\ \hline \text{Ans. } 11\,(x^2-a) \end{array}$$

(4.)

$$\begin{array}{r} 4\sqrt{a-x} \\ 3\sqrt{a-x} \\ -7\sqrt{a-x} \\ 2\sqrt{a-x} \\ \sqrt{a-x} \\ \hline 3\sqrt{a-x} \end{array}$$

(5.)

$$\begin{array}{r} 7\,y-4\,(a+b) \\ 6\,y+2\,(a+b) \\ 2\,y+\,(a+b) \\ y-3\,(a+b) \\ \hline 16\,y-4\,(a+b) \end{array}$$

(6.)

$$\begin{array}{lr} 2\,(x-y)^2 & \\ 3\,(x-y)^2 & \\ (x-y)^2 & \\ -\,(x-y)^2 + & (x+y) \\ & (x+y) \\ \hline 5\,(x-y)^2 + & 2\,(x+y) \end{array}$$

(ART. 53, page 28.)

(2.)

$$\begin{array}{r} b\,x \\ a\,b\,x \\ 2\,c\,x \\ \hline (b+a\,b+2\,c)\,x \end{array}$$

(3.)

$$\begin{array}{r} 3\,a\,y \\ -\,c\,y \\ -\,2\,a\,y \\ \hline (a-c)\,y \end{array}$$

(4.)

$$\begin{array}{r} (a+b)\,x \\ (a-c)\,x \\ \hline (2\,a+b-c)\,x \end{array}$$

(5.)

$$\begin{array}{r} (a+b)\,x \\ 2\,c\,x \\ 2\,x \\ \hline (a+b+2\,c+2)\,x \end{array}$$

(6.)

$$\begin{array}{lr} a\,x+b & \\ c\,x & +\,d \\ \hline (a+c)\,x+b & +\,d \end{array}$$

(7.)

$$\begin{array}{r} a\,x + 7\,m \\ 7\,a\,x - 3\,m \\ b\,x + 4\,m \\ \hline (8\,a + b)\,x + 8\,m \end{array}$$

(8.)

$$\begin{array}{r} a\,x^2 + b\,x \\ c\,x^2 - d\,x \\ \hline (a + c)\,x^2 + (b - d)\,x \end{array}$$

SUBTRACTION.

(ART. 54, pp. 30–32.)

10. Ans. $11\,y$.
11. Ans. $9\,a$.
12. Ans. $3\,a\,x^2$.
13. Ans. $-16\,x^2 y^3$.
19. Ans. 0.
21. Ans. $-15\,d$.

(22.)

$$\begin{array}{r} a + b \\ -a \quad\; \\ \hline 2\,a + b \end{array}$$

(23.)

$$\begin{array}{r} a + b \\ a - b \\ \hline 2\,b \end{array}$$

(24.)

$$\begin{array}{r} a - b \\ a + b \\ \hline -2\,b \end{array}$$

(25.)

$$\begin{array}{r} a - \quad b \\ -a + \quad b \\ \hline 2\,a - 2\,b \end{array}$$

(26.)

$$\begin{array}{l} 5\,x\,y \\ 3\,x\,y - 3 \\ \hline 2\,x\,y + 3 \end{array}$$

(27.)

$$\begin{array}{r} a + b + \quad c \\ a - b - \quad c \\ \hline 2\,b + 2\,c \end{array}$$

(28.)

$$\begin{array}{r} x \quad\; \\ x + y \\ \hline -y \end{array}$$

(29.)

$$\begin{array}{r} x \qquad + 5 \\ y - 2 \\ \hline x - y + 7 \end{array}$$

(32.)

$$\begin{array}{r} 2\,x^2 - y^2 \\ -2\,x^2 - y^2 \\ \hline 4\,x^2 \qquad \end{array}$$

(35.)

$$\begin{array}{r} 5\,x\,y + 2\,x^3 + 2\,a \\ 3\,x\,y - \quad x^3 - 7\,a \\ \hline 2\,x\,y + 3\,x^3 + 9\,a \end{array}$$

(36.)

$$\begin{array}{l} 6\,a\,b\,x + 12 - 3\,x\,y \\ 3\,a\,b\,x - \quad 7 \\ \hline 3\,a\,b\,x + 19 - 3\,x\,y \end{array}$$

(37.)

$$\begin{array}{l} a\,x^3 - \quad b\,x^2 + c\,x \\ \qquad + \quad b\,x^2 + c\,x - 12\,d \\ \hline a\,x^3 - 2\,b\,x^2 \qquad\qquad + 12\,d \end{array}$$

(38.)

$$\begin{array}{r} 3\,a + b + c - d \\ 3\,a + b \qquad\quad - 18 \\ \hline c - d + 18 \end{array}$$

(39.)

$$\begin{array}{l} 5x \qquad\qquad -\ b \\ \quad -2xy+\ b \\ \hline 5x+2xy-2b \end{array}$$

(40.)

$$\begin{array}{r} 3a(a-y)+\ 4by+\ a^3 \\ 2a(a-y)-\ 7by+4a^3 \\ \hline a(a-y)+11by-3a^3 \end{array}$$

(41.)

$$\begin{array}{l} a^3+3b^2c+ab^2-abc \\ \qquad\quad +ab^2-abc+b^2 \\ \hline a^3+3b^2c \qquad\qquad\quad -b^2 \end{array}$$

(42.)

$$\begin{array}{r} -\ 6a-4b-12c+12x \\ 8a+4b-\ 5c-\ 2x \\ \hline -14a-8b-\ 7c+14x \end{array}$$

(43.)

$$\begin{array}{l} 2ab+\ b^2-4c+\ bc \\ 3ab+2b^2-\ c-3bc+4b^2 \\ \hline -ab-\ b^2-3c+4bc-4b^2=-ab-3c+4bc-5b^2 \end{array}$$

(Art. 55, page 33.)

6. $-6a^2-b-4a^2-b^2=-10a^2-b-b^2$.

(Art. 56, p. 34.)

(4.)

$$\begin{array}{r} ay+by \\ cy+by \\ \hline (a-c)y \end{array}$$

(5.)

$$\begin{array}{r} ax-b \\ cx \qquad -d \\ \hline (a-c)x-b+d \end{array}$$

(6.)

$$\begin{array}{r} ay-by+cy \\ ay-by+\ y \\ \hline (c-1)y \end{array}$$

(7.)

$$\begin{array}{r} ax^2 \qquad +bx \\ cx^2 \qquad -cx \\ \hline (a-c)x^2+(b+c)x \end{array}$$

(8.)

$$\begin{array}{llll} a\,x^2 & +m\,x\,y & +n\,x+b & \\ c\,x^2 & -d\,x\,y & +e\,x & -z \\ \hline (a-c)\,x^2 & +(m+d)\,x\,y & +(n-e)\,x & +b+z \end{array}$$

MULTIPLICATION.

(Art. 62, pp. 37, 38.)

7. Ans. $4\,x^2$.
8. Ans. $-34\,a\,b$.
9. Ans. $12\,a\,b$.
10. Ans. $-28\,b\,c$.

(Art. 63, pp. 39, 40.)

5. Ans. $7\,x^3+4\,x^2\,y+a^2\,x^2$.
6. Ans. $-20\,a^2\,b\,x^2+4\,a^3\,x^3-4\,a\,x^3$.

(Art. 64, pp. 41, 42.)

(4.)

$$\begin{array}{rl} & 3\,a^2-2\,y \\ & x+y \\ \hline & 3\,a^2\,x-2\,x\,y \\ & \qquad\qquad +3\,a^2\,y-2\,y^2 \\ \hline \text{Ans.} & 3\,a^2\,x-2\,x\,y+3\,a^2\,y-2\,y^2 \end{array}$$

(5.)

$$\begin{array}{rl} & a-b \\ & a-b \\ \hline & a^2-a\,b \\ & \quad -a\,b+b^2 \\ \hline \text{Ans.} & a^2-2\,a\,b+b^2 \end{array}$$

(6.)

$$\begin{array}{rl} & 5\,a\,x+3\,x \\ & 3\,a\,x+2\,x \\ \hline & 15\,a^2\,x^2+9\,a\,x^2 \\ & \qquad\qquad 10\,a\,x^2+6\,x^2 \\ \hline \text{Ans.} & 15\,a^2\,x^2+19\,a\,x^2+6\,x^2 \end{array}$$

(9.)

$$\begin{array}{l} 3\,x+2\,y \\ 2\,x-3\,y \\ \hline 6\,x^2+4\,x\,y \\ \qquad -9\,x\,y-6\,y^2 \\ \hline 6\,x^2-5\,x\,y-6\,y^2 \end{array}$$

(10.)

$$\begin{array}{l}
5a^2 + 3x \\
5a^2 + 3x \\
\hline
25a^4 + 15a^2x \\
\qquad + 15a^2x + 9x^2 \\
\hline
25a^4 + 30a^2x + 9x^2
\end{array}$$

(11.)

$$\begin{array}{l}
a + 2x \\
a - 3x \\
\hline
a^2 + 2ax \\
\qquad - 3ax - 6x^2 \\
\hline
a^2 - ax - 6x^2
\end{array}$$

(12.)

$$\begin{array}{l}
3a - x \\
2a + 4x \\
\hline
6a^2 - 2ax \\
\qquad + 12ax - 4x^2 \\
\hline
6a^2 + 10ax - 4x^2
\end{array}$$

(13.)

$$\begin{array}{l}
x + y \\
x + y \\
\hline
x^2 + xy \\
\qquad xy + y^2 \\
\hline
x^2 + 2xy + y^2
\end{array}$$

(14.)

$$\begin{array}{l}
x - y \\
x + y \\
\hline
x^2 - xy \\
\qquad xy - y^2 \\
\hline
x^2 \qquad - y^2
\end{array}$$

(15.)

$$\begin{array}{l}
a^2 + ab + b^2 \\
a - b \\
\hline
a^3 + a^2b + ab^2 \\
\quad - a^2b - ab^2 - b^3 \\
\hline
a^3 \qquad\qquad - b^3
\end{array}$$

(16.)

$$\begin{array}{l}
a^2 - a + 1 \\
a + 1 \\
\hline
a^3 - a^2 + a \\
\qquad a^2 - a + 1 \\
\hline
a^3 \qquad\qquad + 1
\end{array}$$

(17.)

$$\begin{array}{l}
x^3 - ax^2 + a^2x - a^3 \\
x + a \\
\hline
x^4 - ax^3 + a^2x^2 - a^3x \\
\qquad ax^3 - a^2x^2 + a^3x - a^4 \\
\hline
x^4 \qquad\qquad\qquad - a^4
\end{array}$$

(18.)

$$\begin{array}{l}
a^4 - a^3y + a^2y^2 - ay^3 + y^4 \\
a + y \\
\hline
a^5 - a^4y + a^3y^2 - a^2y^3 + ay^4 \\
\qquad a^4y - a^3y^2 + a^2y^3 - ay^4 + y^5 \\
\hline
a^5 \qquad\qquad\qquad\qquad + y^5
\end{array}$$

(19.)

$$\begin{array}{l}
x^2 + y \\
x^2 + y \\
\hline
x^4 + x^2y \\
\qquad x^2y + y^2 \\
\hline
x^4 + 2x^2y + y^2
\end{array}$$

(20.)

$$\begin{array}{l}
2ab - 3b^2 \\
3ab + 4b^2 \\
\hline
6a^2b^2 - 9ab^3 \\
\qquad\quad\; 8ab^3 - 12b^4 \\
\hline
6a^2b^2 - \;\; ab^3 - 12b^4
\end{array}$$

(21.)

$$\begin{array}{l}
x^2 + xy - y^2 \\
x - y \\
\hline
x^3 + x^2y - \;\; xy^2 \\
\quad - x^2y - \;\; xy^2 + y^3 \\
\hline
x^3 \qquad\quad - 2xy^2 + y^3
\end{array}$$

(22.)

$$\begin{array}{l}
a^2 - 4a + 16 \\
a + 5 \\
\hline
a^3 - 4a^2 + 16a \\
\qquad 5a^2 - 20a + 80 \\
\hline
a^3 + \;\; a^2 - \;\; 4a + 80
\end{array}$$

(23.)

$$\begin{array}{l}
1 - a + a^2 - a^3 \\
1 + a \\
\hline
1 - a + a^2 - a^3 \\
\qquad a - a^2 + a^3 - a^4 \\
\hline
1 \qquad\qquad\qquad - a^4
\end{array}$$

(24.)

$$\begin{array}{l}
x^2 + xy + y^2 \\
x^2 - xy + y^2 \\
\hline
x^4 + x^3y + x^2y^2 \\
\quad - x^3y - x^2y^2 - xy^3 \\
\qquad\qquad\; x^2y^2 + xy^3 + y^4 \\
\hline
x^4 \qquad + x^2y^2 \qquad + y^4
\end{array}$$

(25.)

$$\begin{array}{l}
a - bx \\
c - dx \\
\hline
ac - bcx \\
\quad - adx + bdx^2 \\
\hline
ac - (bc + ad)x + bdx^2
\end{array}$$

(26.)

$$\begin{array}{l}
3x^2 - \;\; 2xy - y^2 \\
2x - \;\; 4y \\
\hline
6x^3 - \;\; 4x^2y - 2xy^2 \\
\quad - 12x^2y + 8xy^2 + 4y^3 \\
\hline
6x^3 - 16x^2y + 6xy^2 + 4y^3
\end{array}$$

(27.)

$$\begin{array}{l}
x - y + z \\
x + y - z \\
\hline
x^2 - xy + xz \\
\quad + xy \qquad - y^2 + \;\; yz \\
\qquad\qquad - xz \qquad + \;\; yz - z^2 \\
\hline
x^2 \qquad\qquad - y^2 + 2yz - z^2
\end{array}$$

(28.)

$$\begin{array}{l}
27\,x^3 + 9\,x^2\,y + 3\,x\,y^2 + y^3 \\
3\,x - y \\
\hline
81\,x^4 + 27\,x^3\,y + 9\,x^2\,y^2 + 3\,x\,y^3 \\
\qquad - 27\,x^3\,y - 9\,x^2\,y^2 - 3\,x\,y^3 - y^4 \\
\hline
81\,x^4 - y^4
\end{array}$$

(29.)

$$\begin{array}{l}
1 + x + x^4 + x^5 \\
1 - x + x^2 - x^3 \\
\hline
1 + x + x^4 + x^5 \\
\quad - x - x^2 - x^5 - x^6 \\
\qquad x^2 + x^3 + x^6 + x^7 \\
\qquad\quad - x^3 - x^4 - x^7 - x^8 \\
\hline
1 - x^8
\end{array}$$

(30.)

$$\begin{array}{l}
a^n + y^n \\
a^n + y^n \\
\hline
a^{2n} + a^n\,y^n \\
\qquad a^n\,y^n + y^{2n} \\
\hline
a^{2n} + 2\,a^n\,y^n + y^{2n}
\end{array}$$

(Art. 65, pp. 42, 43.)

(1.)

$$\begin{array}{l}
a - b \\
a - b \\
\hline
a^2 - a\,b \\
\quad - a\,b + b^2 \\
\hline
a^2 - 2\,a\,b + b^2
\end{array}$$

(2.)

$$\begin{array}{l}
a + b \\
c + d \\
\hline
a\,c + b\,c \\
\qquad + a\,d + b\,d \\
\hline
a\,c + b\,c + a\,d + b\,d
\end{array}$$

(3.)

$$\begin{array}{l}
a + b \\
a + b \\
\hline
a^2 + ab \\
\quad + ab + b^2 \\
\hline
a^2 + 2ab + b^2 \\
a + b \\
\hline
a^3 + 2a^2b + ab^2 \\
\qquad a^2b + 2ab^2 + b^3 \\
\hline
a^3 + 3a^2b + 3ab^2 + b^3
\end{array}$$

(4.)

$$\begin{array}{l}
a + b \\
a + b \\
\hline
a^2 + ab \\
\quad + ab + b^2 \\
\hline
a^2 + 2ab + b^2 \\
a - b \\
\hline
a^3 + 2a^2b + ab^2 \\
\quad - a^2b - 2ab^2 - b^3 \\
\hline
a^3 + a^2b - ab^2 - b^3
\end{array}$$

(5.)

$$\begin{array}{l}
x^2 - xy + y^2 \\
x + y \\
\hline
x^3 - x^2y + xy^2 \\
\qquad x^2y - xy^2 + y^3 \\
\hline
x^3 + y^3
\end{array}$$

(6.)

$$\begin{array}{l}
a + b + c \\
a - b - c \\
\hline
a^2 + ab + ac \\
\quad - ab - b^2 - bc \\
\quad - ac - bc - c^2 \\
\hline
a^2 - b^2 - 2bc - c^2
\end{array}$$

(7.)

$$\begin{array}{l}
2x + 3 \\
2x - 3 \\
\hline
4x^2 + 6x \\
\quad - 6x - 9 \\
\hline
4x^2 - 9 \\
4x^2 + 9 \\
\hline
16x^4 - 36x^2 \\
\qquad 36x^2 - 81 \\
\hline
16x^4 - 81
\end{array}$$

(8.)

$$\begin{array}{l}
a^n + b^n \\
a - b \\
\hline
a^{n+1} + ab^n \\
\qquad - a^nb - b^{n+1} \\
\hline
a^{n+1} + ab^n - a^nb - b^{n+1}
\end{array}$$

(9.)

$$
\begin{array}{l}
4\,a^{m} + 6\,b^{n} \\
\quad a^{m} - \quad b^{3} \\
\hline
4\,a^{2m} + 6\,a^{m}\,b^{n} \\
\qquad\qquad\qquad - 4\,a^{m}\,b^{3} - 6\,b^{n+3} \\
\hline
4\,a^{2m} + 6\,a^{m}\,b^{n} - 4\,a^{m}\,b^{3} - 6\,b^{n+3}
\end{array}
$$

DIVISION.

(Art. 72, page 47.)

6. $\dfrac{-16\,a^{3}\,b^{2}}{-8\,a^{5}\,b} = \dfrac{2\,b}{a^{2}} = 2\,a^{-2}\,b.$

8. Ans. $7\,m$.

11. Ans. $-3\,a$.

18. $\dfrac{36\,a^{7}\,b^{6}\,c^{2}}{9\,a^{5}\,b^{2}\,c^{4}} = \dfrac{4\,a^{2}\,b^{4}}{c^{2}} = 4\,a^{2}\,b^{4}\,c^{-2}.$

21. $\dfrac{27\,a\,b^{2}\,(x+y)^{3}}{3\,b^{3}\,(x+y)^{5}} = \dfrac{9\,a}{b\,(x+y)^{2}} = 9\,a\,b^{-1}\,(x+y)^{-2}.$

The negative exponents found in the above answers may be obtained at once by subtracting the exponents of the divisor from the corresponding ones of the dividend; or, they may be obtained, as above, by transferring the remaining factors of the divisor to the dividend, the sign of their exponents being changed.

(Art. 73, pp. 48, 49.)

5. Ans. $3\,a^{2}\,b + 5\,a\,b - 2\,c$.

6. Ans. $5 + 10\,x - 12\,b^{2}$.

16. $3\,a + (a + b) = 4\,a + b.$

17. $4 - (x + 3) = 4 - x - 3 = 1 - x.$

18. The exponent of a in $5\,a\,b^2\,x$ is 1, which, subtracted from -2, gives -3 as the exponent of a in the first term of the quotient. The exponent -1, of $2\,b^{-1}$, is obtained in a similar manner, by subtracting the exponent 2 from the exponent 1. The answer may also be thus expressed: $3\,\frac{b}{a^3} - \frac{2}{b} - 1.$

(ART. 74, pp. 52, 53.)

(4.)

$$
\begin{array}{r|l}
a + b) & a^2 - b^2\ (a - b \\
 & \underline{a^2 + a\,b} \\
 & - a\,b - b^2 \\
 & \underline{- a\,b - b^2}
\end{array}
$$

(5.)

$$
\begin{array}{l|l}
a^3 + 2\,a^2\,b + 2\,a\,b^2 + b^3 & a^2 + a\,b + b^2 \\ \cline{2-2}
\underline{a^3 + a^2\,b + a\,b^2} & a + b \\
a^2\,b + a\,b^2 + b^3 & \\
\underline{a^2\,b + a\,b^2 + b^3} &
\end{array}
$$

(6.)

$$
\begin{array}{l|l}
a^3 - 3\,a^2\,b + 3\,a\,b^2 - b^3 & a - b \\ \cline{2-2}
\underline{a^3 - a^2\,b} & a^2 - 2\,a\,b + b^2 \\
- 2\,a^2\,b + 3\,a\,b^2 & \\
\underline{- 2\,a^2\,b + 2\,a\,b^2} & \\
a\,b^2 - b^3 & \\
\underline{a\,b^2 - b^3} &
\end{array}
$$

(7.)

$$\begin{array}{l}
a - 1)\ a^3 - 1\ \ (a^2 + a + 1 \\
 \underline{a^3 - a^2} \\
 a^2 - 1 \\
 \underline{a^2 - a} \\
 a - 1 \\
 \underline{a - 1}
\end{array}$$

(8.)

$$\begin{array}{l|l}
8\,a^3 - 4\,a^2\,b - 6\,a\,b^2 + 3\,b^3 & 2\,a - b \\ \cline{2-2}
8\,a^3 - 4\,a^2\,b & 4\,a^2 - 3\,b^2 \\ \cline{1-1}
 - 6\,a\,b^2 + 3\,b^3 & \\
 \underline{- 6\,a\,b^2 + 3\,b^3} &
\end{array}$$

(9.)

$$\begin{array}{l}
x^2 - 2\,x + 3)\ x^4 + 4\,x + 3\quad (x^2 + 2\,x + 1 \\
 \underline{x^4 - 2\,x^3 + 3\,x^2} \\
 2\,x^3 - 3\,x^2 + 4\,x \\
 \underline{2\,x^3 - 4\,x^2 + 6\,x} \\
 x^2 - 2\,x + 3 \\
 \underline{x^2 - 2\,x + 3}
\end{array}$$

(10.)

$$\begin{array}{l}
a^2 + a\,x + x^2)\ a^3 - x^3\qquad\qquad (a - x \\
 \underline{a^3 + a^2\,x + a\,x^2} \\
 - a^2\,x - a\,x^2 - x^3 \\
 \underline{- a^2\,x - a\,x^2 - x^3}
\end{array}$$

(11.)

$$\begin{array}{l}
a^2 - 2ab + 4b^2)\ a^4 + 4a^2b^2 + 16b^4\ (a^2 + 2ab + 4b^2 \\
\qquad\qquad\qquad\ \ \underline{a^4 - 2a^3b + 4a^2b^2} \\
\qquad\qquad\qquad\qquad\quad 2a^3b + 16b^4 \\
\qquad\qquad\qquad\qquad\quad \underline{2a^3b - 4a^2b^2 + 8ab^3} \\
\qquad\qquad\qquad\qquad\qquad\quad 4a^2b^2 - 8ab^3 + 16b^4 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{4a^2b^2 - 8ab^3 + 16b^4}
\end{array}$$

(12.)

$$\begin{array}{l}
a - x)\ a^3 - x^3\ (a^2 + ax + x^2 \\
\qquad\quad \underline{a^3 - a^2x} \\
\qquad\qquad\ a^2x - x^3 \\
\qquad\qquad\ \underline{a^2x - ax^2} \\
\qquad\qquad\qquad\ ax^2 - x^3 \\
\qquad\qquad\qquad\ \underline{ax^2 - x^3}
\end{array}$$

(13.)

$$\begin{array}{l}
a + x)\ a^3 + x^3\ (a^2 - ax + x^2 \\
\qquad\quad \underline{a^3 + a^2x} \\
\qquad\quad\ - a^2x + x^3 \\
\qquad\quad\ \underline{- a^2x - ax^2} \\
\qquad\qquad\qquad ax^2 + x^3 \\
\qquad\qquad\qquad \underline{ax^2 + x^3}
\end{array}$$

(14.)

$$\begin{array}{l|l}
x^3 - 5x^2 - 46x - 40 & x + 4 \\ \cline{2-2}
\underline{x^3 + 4x^2 \qquad\qquad\qquad} & x^2 - 9x - 10 \\
\quad - 9x^2 - 46x & \\
\quad \underline{- 9x^2 - 36x} & \\
\qquad\qquad - 10x - 40 & \\
\qquad\qquad \underline{- 10x - 40} &
\end{array}$$

(15.)

$$
\begin{array}{l|l}
x^7 - 1 & x - 1 \\
\cline{2-2}
x^7 - x^6 & x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\hline
\quad x^6 - 1 & \\
\quad x^6 - x^5 & \\
\quad\quad x^5 - 1 & \\
\quad\quad x^5 - x^4 & \\
\quad\quad\quad x^4 - 1 & \\
\quad\quad\quad x^4 - x^3 & \\
\quad\quad\quad\quad x^3 - 1 & \\
\quad\quad\quad\quad x^3 - x^2 & \\
\quad\quad\quad\quad\quad x^2 - 1 & \\
\quad\quad\quad\quad\quad x^2 - x & \\
\quad\quad\quad\quad\quad\quad x - 1 & \\
\quad\quad\quad\quad\quad\quad x - 1 &
\end{array}
$$

(16.)

$$
\begin{array}{l}
a + x)\ a^5 + x^5 \quad (a^4 - a^3 x + a^2 x^2 - a x^3 + x^4 \\
\qquad a^5 + a^4 x \\
\hline
\qquad - a^4 x + x^5 \\
\qquad - a^4 x - a^3 x^2 \\
\hline
\qquad\qquad a^3 x^2 + x^5 \\
\qquad\qquad a^3 x^2 + a^2 x^3 \\
\hline
\qquad\qquad\quad - a^2 x^3 + x^5 \\
\qquad\qquad\quad - a^2 x^3 - a x^4 \\
\hline
\qquad\qquad\qquad a x^4 + x^5 \\
\qquad\qquad\qquad a x^4 + x^5 \\
\hline
\end{array}
$$

(17.)

$$
\begin{array}{l|l}
21\,a^5 - 21\,b^5 & 7\,a - 7\,b \\ \cline{2-2}
21\,a^5 - 21\,a^4\,b & 3\,a^4 + 3\,a^3\,b + 3\,a^2\,b^2 + 3\,a\,b^3 + 3\,b^4 \\ \cline{1-1}
\qquad 21\,a^4\,b - 21\,b^5 & \\
\qquad 21\,a^4\,b - 21\,a^3\,b^2 & \\ \cline{1-1}
\qquad\qquad 21\,a^3\,b^2 - 21\,b^5 & \\
\qquad\qquad 21\,a^3\,b^2 - 21\,a^2\,b^3 & \\ \cline{1-1}
\qquad\qquad\qquad 21\,a^2\,b^3 - 21\,b^5 & \\
\qquad\qquad\qquad 21\,a^2\,b^3 - 21\,a\,b^4 & \\ \cline{1-1}
\qquad\qquad\qquad\qquad 21\,a\,b^4 - 21\,b^5 & \\
\qquad\qquad\qquad\qquad 21\,a\,b^4 - 21\,b^5 & \\ \cline{1-1}
\end{array}
$$

(18.)

$$
\begin{array}{l|l}
2\,a^4 + 2\,a^3\,b + 5\,a^2\,b^2 - 6\,a\,b^3 + 4\,b^4 & 2\,a^2 - 2\,a\,b + b^2 \\ \cline{2-2}
2\,a^4 - 2\,a^3\,b + \quad a^2\,b^2 & a^2 + 2\,a\,b + 4\,b^2 \\ \cline{1-1}
\qquad 4\,a^3\,b + 4\,a^2\,b^2 - 6\,a\,b^3 & \\
\qquad 4\,a^3\,b - 4\,a^2\,b^2 + 2\,a\,b^3 & \\ \cline{1-1}
\qquad\qquad 8\,a^2\,b^2 - 8\,a\,b^3 + 4\,b^4 & \\
\qquad\qquad 8\,a^2\,b^2 - 8\,a\,b^3 + 4\,b^4 & \\ \cline{1-1}
\end{array}
$$

(19.)

$$
\begin{array}{l|l}
2\,a^{m+1} - 2\,a^{n+1} - a^{m+n} + a^{2n} & 2\,a - a^n \\ \cline{2-2}
2\,a^{m+1} \qquad\qquad\quad - a^{m+n} & a^m - a^n \\ \cline{1-1}
\qquad - 2\,a^{n+1} \qquad\qquad + a^{2n} & \\
\qquad - 2\,a^{n+1} \qquad\qquad + a^{2n} & \\ \cline{1-1}
\end{array}
$$

(20.)

$$
\begin{array}{l}
a + x)\ a^4 - 3x^4\ \left(a^3 - a^2x + ax^2 - x^3 - \dfrac{2x^4}{a+x}\right. \\
\qquad\quad \underline{a^4 + a^3x} \\
\qquad\qquad -a^3x - 3x^4 \\
\qquad\qquad \underline{-a^3x - a^2x^2} \\
\qquad\qquad\qquad\quad a^2x^2 - 3x^4 \\
\qquad\qquad\qquad\quad \underline{a^2x^2 + ax^3} \\
\qquad\qquad\qquad\qquad\quad -ax^3 - 3x^4 \\
\qquad\qquad\qquad\qquad\quad \underline{-ax^3 - x^4} \\
\qquad\qquad\qquad\qquad\qquad\quad -2x^4
\end{array}
$$

(21.)

$$
\begin{array}{l|l}
a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5 & a^2 - 2ab + b^2 \\ \cline{2-2}
\underline{a^5 - 2a^4b + a^3b^2} & a^3 - 3a^2b + 3ab^2 - b^3 \\
\quad -3a^4b + 9a^3b^2 - 10a^2b^3 & \\
\quad \underline{-3a^4b + 6a^3b^2 - 3a^2b^3} & \\
\qquad\qquad 3a^3b^2 - 7a^2b^3 + 5ab^4 & \\
\qquad\qquad \underline{3a^3b^2 - 6a^2b^3 + 3ab^4} & \\
\qquad\qquad\qquad - a^2b^3 + 2ab^4 - b^5 & \\
\qquad\qquad\qquad \underline{- a^2b^3 + 2ab^4 - b^5} &
\end{array}
$$

THEOREMS.

TEACHERS will observe that the principles of Involution, for the *second power* only, are here anticipated to some extent. They will also readily perceive that it is important to introduce these principles at this early stage, on account of their use in the operations connected with factoring and fractions.

The pupil already understands that a quantity is *squared*

when it is multiplied by itself (Art. 20); and that, so far as the literal part of a monomial is concerned, this is effected by adding the exponent of each letter to itself (Art. 62), or, which is the same thing, multiplying it by 2.

THEOREM I.

(Art. 76, page 54.)

2. $(2x+y)^2 = (2x)^2 + 2(2x)y + y^2 = 4x^2 + 4xy + y^2$.

3. $(6a^2 + 2a^2b)^2 = (6a^2)^2 + 2(6a^2)(2a^2b) + (2a^2b)^2$
$= 36a^4 + 24a^4b + 4a^4b^2$.

4. $(a^3b^2 + 3a^2b^3c^4)^2 = (a^3b^2)^2 + 2(a^3b^2)(3a^2b^3c^4)$
$+ (3a^2b^3c^4)^2 = a^6b^4 + 6a^5b^5c^4 + 9a^4b^6c^8$.

THEOREM II.

(Art. 77, p. 54.)

2. $(5c-1)^2 = (5c)^2 - 2(5c) + 1 = 25c^2 - 10c + 1$.

3. $(a^2-b^2)^2 = (a^2)^2 - 2a^2b^2 + (b^2)^2 = a^4 - 2a^2b^2 + b^4$.

4. $(5a^2b^2 - 10a^2b^3)^2 = (5a^2b^2)^2 - 2(5a^2b^2)(10a^2b^3)$
$+ (10a^2b^3)^2 = 25a^4b^4 - 100a^4b^5 + 100a^4b^6$.

THEOREM III.

(Art. 78, p. 55.)

1. $(3a+2b)(3a-2b) = (3a)^2 - (2b)^2 = 9a^2 - 4b^2$.

3. $(5a+b)(5a-b) = (5a)^2 - b^2 = 25a^2 - b^2$.

4. $(9x+1)(9x-1) = (9x)^2 - 1 = 81x^2 - 1$.

5. $(3a^2c+10ab^3)(3a^2c-10ab^3) = (3a^2c)^2 - (10ab^3)^2$
$= 9a^4c^2 - 100a^2b^6$.

6 $(3x^2y+12xy^3)(3x^2y-12xy^3) = (3x^2y)^2 - (12xy^3)^2$
$= 9x^4y^2 - 144x^2y^6$

MISCELLANEOUS EXAMPLES.

(pp. 55, 56.)

2. $(3\,a - 2)^2 = (3\,a)^2 - 2\,(3\,a)\,2 + 2^2 = 9\,a^2 - 12\,a + 4.$

3. $(9\,a\,b + 2\,b^2)^2 = (9\,a\,b)^2 + 2\,(9\,a\,b)\,(2\,b^2) + (2\,b^2)^2$
$= 81\,a^2\,b^2 + 36\,a\,b^3 + 4\,b^4.$

5. $(2 - x^m)^2 = 2^2 - 2\,(2\,x^m) + (x^m)^2 = 4 - 4\,x^m + x^{2m}.$

6. $(a^3 + 1)\,(a^3 - 1) = (a^3)^2 - 1 = a^6 - 1.$

7. $(a^2 - b^2)\,(a^2 + b^2) = (a^2)^2 - (b^2)^2 = a^4 - b^4.$

8. $(1 - 3\,c^2)^2 = 1 - 2\,(3\,c^2) + (3\,c^2)^2 = 1 - 6\,c^2 + 9\,c^4.$

9. $(2 + \overline{a - b})\,(2 - \overline{a - b}) = 2^2 - \overline{a - b}^2 = 4 - (a - b)^2.$

10. $2\,(a + b)\,(a - b) = 2\,(a^2 - b^2) = 2\,a^2 - 2\,b^2.$

11. $3^3\,(x^2 - a^2)^2 = 3^3\,(x^2)^2 - (3^3)\,2\,a^2\,x^2 + 3^3\,(a^2)^2$
$= 27\,x^4 - 54\,a^2\,x^2 + 27\,a^4.$

12. $(1 - 4\,a)\,(1 - 4\,a) = 1 - 2\,(4\,a) + (4\,a)^2$
$= 1 - 8\,a + 16\,a^2.$

13. $(3\,m + 4\,n)\,(3\,m - 4\,n) = (3\,m)^2 - (4\,n)^2$
$= 9\,m^2 - 16\,n^2.$

14. $(3\,a - 4\,x)\,(3\,a + 4\,x) = (3\,a)^2 - (4\,x)^2$
$= 9\,a^2 - 16\,x^2$, Ans.

15. $(2\,a + 3\,x)\,(2\,a + 3\,x) = (2\,a)^2 + 2\,(2\,a)\,(3\,x)$
$+ (3\,x)^2 = 4\,a^2 + 12\,a\,x + 9\,x^2.$

16. $(2\,a\,c - 3\,b\,c)\,(2\,a\,c - 3\,b\,c) = (2\,a\,c)^2 - 2\,(2\,a\,c)\,(3\,b\,c)$
$+ (3\,b\,c)^2 = 4\,a^2\,c^2 - 12\,a\,b\,c^2 + 9\,b^2\,c^2.$

17. $(3 - \overline{a + b})\,(3 - \overline{a + b}) = 3^2 - 2\,(3)\,(a + b)$
$+ (a + b)^2 = 9 - 6\,(a + b) + (a + b)^2.$

18. $(5\,a^2\,b^2 + 7\,a\,b)\,(5\,a^2\,b^2 - 7\,a\,b) = (5\,a^2\,b^2)^2 - (7\,a\,b)^2$
$= 25\,a^4\,b^4 - 49\,a^2\,b^2.$

19. $(x + a)(x - a)(x^2 - a^2) = (x^2 - a^2)(x^2 - a^2)$
$= x^4 - 2a^2x^2 + a^4.$

20. $(x + 2)(x - 2)(x - 3)(x + 3) = (x^2 - 4)(x^2 - 9)$
$= x^4 - 13x^2 + 36.$

21. $(2x + 3)(2x - 3)(4x^2 + 9) = (4x^2 - 9)(4x^2 + 9)$
$= 16x^4 - 81.$

22. $(x^2 - 1)(x^2 + 1)(x^4 - 1) = (x^4 - 1)(x^4 - 1)$
$= x^8 - 2x^4 + 1.$

FACTORING.

(Art. 88, page 59.)

3. Ans. $3 \times 7\, m\, m\, m\, n\, n\, x.$

5. Ans. $2 \times 2 \times 2 \times 7\, a\, a\, a\, b\, b\, b\, b\, c\, c\, x\, x\, x\, y.$

(Art. 89, page 60.)

4. Ans. $x(a + 1).$

6. Ans. $7bc^2x(2c^2 - 3bc + 1).$

(Art. 90, page 61.)

3. $4a^2 + 12ab + 9b^2 = (2a)^2 + 2(2a)(3b) + (3b)^2$
$= (2a + 3b)^2 = (2a + 3b)(2a + 3b).$

4. Ans. $(2a - 3b)(2a - 3b).$

5. $a^2 - 4ab^2 + 4b^4 = a^2 - 2a(2b^2) + (2b^2)^2 = (a - 2b^2)^2.$

7. $1 + 2x^2 + x^4 = 1 + 2x^2 + (x^2)^2 = (1 + x^2)^2.$
Ans. $(1 + x)(1 + x).$

8. $4x^2 + 4xy + y^2 = (2x)^2 + 2(2x)y + y^2 = (2x + y)^2.$

9. $25m^4 + 10m^2n + n^2 = (5m^2)^2 + 2(5m^2)n + n^2$
$= (5m^2 + n)(5m^2 + n).$

(ART. 91, page 62.)

3. Ans. $(x + y)\ (x - y)$.

4. $4\,x^2 - y^2 = (2\,x)^2 - y^2 = (2\,x + y)\ (2\,x - y)$.

5. $9\,a^2 - 4\,b^2 = (3\,a)^2 - (2\,b)^2 = (3\,a + 2\,b)\ (3\,a - 2\,b)$.

6. $64\,a^2\,b^2 - 16\,c^2\,d^2 = (8\,a\,b)^2 - (4\,c\,d)^2$.

Ans. $(8\,a\,b + 4\,c\,d)\ (8\,a\,b - 4\,c\,d)$.

7. $1 - 81\,x^2 = 1 - (9\,x)^2 = (1 + 9\,x)\ (1 - 9\,x)$.

8. $c^6 - a^4\,y^2 = (c^3)^2 - (a^2\,y)^2 = (c^3 + a^2\,y)\ (c^3 - a^2\,y)$.

10. $1 - c^4 = (1 + c^2)\ (1 - c^2) = (1 + c^2)\ (1 + c)\ (1 - c)$.

11. $16\,y^8 - 1 = (4\,y^4 + 1)\ (4\,y^4 - 1) = (4\,y^4 + 1)$ $(2\,y^2 + 1)\ (2\,y^2 - 1)$.

12. $a^8 - c^8 = (a^4 + c^4)\ (a^4 - c^4) = (a^4 + c^4)\ (a^2 + c^2)$ $(a^2 - c^2) = (a^4 + c^4)\ (a^2 + c^2)\ (a + c)\ (a - c)$.

13. $x^8 - y^4 = (x^4 + y^2)\ (x^4 - y^2) = (x^4 + y^2)\ (x^2 + y)$ $(x^2 - y)$.

(ART. 92, page 63.)

2. See Art. 87.

3. Using m for a and n for b, we obtain the first answer from Art. 85, and the second from Art. 86. The same results may also be obtained by actually dividing $m^4 - n^4$ by $m - n$ and by $m + n$.

Also $\quad m^3 + m^2\,n + m\,n^2 + n^3 = (m + n)\ (m^2 + n^2)$,

and $\quad m^3 - m^2\,n + m\,n^2 - n^3 = (m - n)\ (m^2 + n^2)$;

hence, $\quad m^4 - n^4 = (m - n)\ (m + n)\ (m^2 + n^2)$.

4. Substituting 1 for a and x for b, the quantity $1 - x^4$ is factored precisely as in the last example. If the second factors be again separated, we obtain the prime factors, thus, $1 - x^4 = (1 - x)(1 + x)(1 + x^2)$.

5. $8x^3 - y^3 = (2x)^3 - y^3 = (2x - y)(4x^2 + 2xy + y^2)$, by Art. 85, or by actual division.

6. $8x^3 + 1 = (2x)^3 + 1 = (2x + 1)(4x^2 - 2x + 1)$, by Art. 87, or by actual division.

7. See Art. 87.

8. The first form of the answer is obtained by Art. 91; this is changed to the second form by Art. 85, and to the third by Art. 87. This latter contains the *prime* factors.

By means of Arts. 86 and 85, or by actual division, we can also obtain

$$a^6 - b^6 = (a + b)(a^5 - a^4 b + a^3 b^2 - a^2 b^3 + a b^4 - b^5),$$

and $a^6 - b^6 = (a - b)(a^5 + a^4 b + a^3 b^2 + a^2 b^3 + a b^4 + b^5)$.

The last factor of each expression is capable of being separated into three others, thus making up the four prime factors,

$$(a + b)(a - b)(a^2 + ab + b^2)(a^2 - ab + b^2).$$

(Art. **93**, p. 64.)

4. $6x^3 + 12x^2 y + 6xy^2 = 6x(x^2 + 2xy + y^2)$
$= 6x(x + y)(x + y)$.

6. $ac - bd + bc - ad = a(c - d) + b(c - d)$
$= (a + b)(c - d)$.

7. $6ax - 2by + 3bx - 4ay = 2a(3x - 2y)$
$+ b(3x - 2y) = (2a + b)(3x - 2y)$.

GREATEST COMMON DIVISOR.

(Art. 98, page 67.)

The examples of Case I. are readily solved mentally.

4. Ans. $5\,a\,b\,c\,d^2$.

(Art. 99, page 70.)

4. According to Note 3, we suppress the monomial factor 3 in $3\,x^3 - 24\,x - 9$, and 2 in $2\,x^3 - 16\,x - 6$, when each becomes $x^3 - 8\,x - 3$, which is therefore their greatest common divisor.

5. According to Note 4, we must multiply the dividend by some quantity, not a factor of the divisor, which will render its first term divisible by the first term of the divisor. As the two expressions are of the *same degree*, it is immaterial which is made the divisor (Note 1).

$$
\begin{array}{r|l}
6\,a^2 + 7\,a\,x - 3\,x^2 & 4\,a^2 - 4\,a\,x - 15\,x^2 \\
2 & (3 \\
\hline
12\,a^2 + 14\,a\,x - 6\,x^2 & \\
12\,a^2 - 12\,a\,x - 45\,x^2 & \\
\hline
26\,a\,x + 39\,x^2 &
\end{array}
$$

$$
\begin{array}{r}
2\,a + 3\,x)\;4\,a^2 - 4\,a\,x - 15\,x^2\,(2\,a - 5\,x \\
4\,a^2 + 6\,a\,x \qquad\qquad \\
\hline
-10\,a\,x - 15\,x^2 \\
-10\,a\,x - 15\,x^2 \\
\hline
\end{array}
$$

The factor $13\,x$ is suppressed in the first remainder (Note 3), and we thus obtain $2\,a + 3\,x$, which proves to be the greatest common divisor

(6.)

$$
\begin{array}{l}
a^2-2ab-3b^2)\ 2a^2+ab-b^2\ (2 \\
\qquad\qquad\qquad\ \ \underline{2a^2-4ab-6b^2} \\
\qquad\qquad\qquad\qquad\quad 5ab+5b^2
\end{array}
$$

$$
\begin{array}{l}
a+b)\ a^2-2ab-3b^2\ (a-3b \\
\qquad\ \ \underline{a^2+ab} \\
\qquad\ \ -3ab-3b^2 \\
\qquad\ \ \underline{-3ab-3b^2}
\end{array}
$$

$$
\begin{array}{l}
a+b)\ 3ac+3bc\ (3c \\
\qquad\ \ \underline{3ac+3bc}
\end{array}
$$

We find the greatest common divisor of the first two given quantities (Note 6), which is $a+b$, and then find the greatest common divisor of $a+b$ and the third given quantity.

(7.)

$$
\begin{array}{l}
x^3+3x^2-4x)\ 2x^4-7x^3+5x^2\ (2x \\
\qquad\qquad\qquad\ \ \underline{2x^4+6x^3-8x^2} \\
\qquad\qquad\qquad\quad -13x^3+13x^2
\end{array}
$$

$$
\begin{array}{l}
x^2-x)\ x^3+3x^2-4x\ (x+4 \\
\qquad\quad \underline{x^3-x^2} \\
\qquad\qquad\ 4x^2-4x \\
\qquad\qquad\ \underline{4x^2-4x}
\end{array}
$$

The factor $13x$ is suppressed in the first remainder, according to Note 3, and the signs are changed, according to Note 5. The factor $13x^2$ is found in both terms of the remainder; but a portion of this, x, is also found in the divisor, and we suppress only that part which is not common to the divisor and remainder.

We might have suppressed x from the original dividend, as we have done in the remainder; or, we might suppress x^2 from one and x from the other, if we finally restore the common factor x to the greatest common divisor (Note 2).

The first division, as given above, might be continued; for the first term of the divisor, x^3, is contained — 13 times in the first term of the remainder, $-13x^3$. The result would, however, be the same, after suppressing the factor found in the remainder.

(ART. **100**, page 71.)

2. $$ab + b^2 = b(a + b),$$
$$ac^2 + bc^2 = c^2(a + b).$$

Hence $a + b$, it being the only common factor, is the greatest common divisor. (Art. 96.)

3. $$a^2 - 2a = a(a - 2),$$
$$ab - 2b = b(a - 2).$$

Hence $a - 2$ is the greatest common divisor.

4. $$a^5 - a^3b^2 = a^3(a^2 - b^2),$$
$$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2).$$

Hence $a^2 - b^2$ is the greatest common divisor.

5. $$ab + am + bn + mn = a(b + m) + n(b + m)$$
$$= (a + n)(b + m),$$
$$b^2n - m^2n = n(b^2 - m^2) = n(b - m)(b + m).$$

Hence $b + m$ is the greatest common divisor.

6. $$a^2 + 2ab + b^2 = (a + b)(a + b),$$
$$a^3 - ab^2 = a(a^2 - b^2) = a(a - b)(a + b).$$

Hence $a + b$ is the greatest common divisor.

7. $$3x^2 - 3y^2 = 3(x^2 - y^2) = 3(x - y)(x + y),$$
$$3x^2 + 6xy + 3y^2 = 3(x^2 + 2xy + y^2) = 3(x + y)(x + y),$$
$$3abx + 3aby = 3ab(x + y).$$

Hence $3(x + y)$, the product of the common factors, is the greatest common divisor (Art. 96).

LEAST COMMMON MULTIPLE.

(Art. **105**, pp. 73, 74.)

4 $(x^2 - x - 12) \div (x + 3) = x - 4,$

and $x^3 + 6x^2 + 9x = x(x^2 + 6x + 9) = x(x + 3)^2$;

hence $\frac{(x^2 - x - 12)(x^3 + 6x^2 + 9x)}{x + 3} = x(x-4)(x+3)^2.$

(5.)

$$\begin{array}{r}
9x^3y^5 = 3^2 x^3 y^5 \\
15xy^2 = 3 \times 5xy^2 \\
18x^5y^6 = 2 \times 3^2 x^5 y^6 \\
\hline
90x^5y^6 = 2 \times 3^2 \times 5x^5y^6
\end{array}$$

(6.)

$$\begin{array}{r}
4abc^2 = 2^2abc^2 \\
6a^2c^3 = 2 \times 3a^2c^3 \\
9ab^5d = 3^2ab^5d \\
\hline
\text{Ans. } 36a^2b^5c^3d = 2^2 \times 3^2a^2b^5c^3d
\end{array}$$

(7.)

$$\begin{array}{r}
5a^3b^2 = 5a^3b^2 \\
10a^2c^2(a + b) = 2 \times 5a^2c^2(a + b) \\
\hline
10a^3b^2c^2(a + b) = 2 \times 5a^3b^2c^2(a + b)
\end{array}$$

(8.)

$$\begin{array}{r}
ab(x + y) = ab(x + y) \\
ac^2(x^3 + y^3) = ac^2(x + y)(x^2 - xy + y^2) \\
\hline
abc^2(x^3 + y^3) = abc^2(x + y)(x^2 - xy + y^2)
\end{array}$$

(9.)

$$\begin{array}{r}
3a + 1 = 3a + 1 \\
3(9a^2 - 1) = 3(3a + 1)(3a - 1) \\
\hline
3(9a^2 - 1) = 3(3a + 1)(3a - 1)
\end{array}$$

(10.)

$$\begin{array}{l} 1+a=1+a \\ 1-a=1-a \\ 1-a^2=(1+a)\,(1-a) \\ \hline 1-a^2=(1+a)\,(1-a) \end{array}$$

(11.)

$a\,(x+y)\,(x-y)=a\,(x^2-y^2)$. See Note 1.

(12.)

$$\begin{array}{r} 3\,a^2\,x+6\,a\,b\,x+3\,b^2\,x=3\,x\,(a+b)^2 \\ 12\,a^2-12\,a\,b+3\,b^2=3\,(2\,a-b)^2 \\ \hline 3\,x\,(a+b)^2\,(2\,a-b)^2 \end{array}$$

FRACTIONS.

REDUCTION.

(ART. 124, page 79.)

5. Ans. $\frac{8\,x}{8}$.

8. $\frac{a^3-a\,b^2}{a^2+2\,a\,b+b^2}=\frac{a\,(a-b)\,(a+b)}{(a+b)\,(a+b)}=\frac{a\,(a-b)}{a+b}=\frac{a^2-a\,b}{a+b}$.

9. $\frac{x^2-1}{2\,x\,y+2\,y}=\frac{(x-1)\,(x+1)}{2\,y\,(x+1)}=\frac{x-1}{2\,y}$.

10. $\frac{a\,x+x^2}{a\,c^2+c^2\,x}=\frac{x\,(a+x)}{c^2\,(a+x)}=\frac{x}{c^2}$.

11. $\frac{x^3-a^2\,x}{x^2+2\,a\,x+a^2}=\frac{x\,(x-a)\,(x+a)}{(x+a)\,(x+a)}=\frac{x\,(x-a)}{x+a}$.

12. $\dfrac{a^5 - a^3 b^2}{a^4 - b^4} = \dfrac{a^3 (a^2 - b^2)}{(a^2 + b^2)(a^2 - b^2)} = \dfrac{a^3}{a^2 + b^2}.$

13. $\dfrac{5a^5 + 10a^4x + 5a^3x^2}{a^4x + 2a^3x^2 + 2ax^3 + x^4} = \dfrac{5a^3(a + x)(a + x)}{(a + x)(a^2x + ax^2 + x^3)}$

$$= \frac{5a^3(a + x)}{a^2x + ax^2 + x^3} = \frac{5a^4 + 5a^3x}{a^2x + ax^2 + x^3}.$$

After factoring the numerator, it is evident that if the two terms of the fraction have any common factor, $a + x$ must be that factor; and we find by actual division that it is contained in the denominator, giving a quotient $a^2x + ax^2 + x^3$.

The last example can readily be solved by finding the greatest common divisor of the terms, according to Art. 99; for the factors cannot readily be determined by inspection, especially in the denominator. Suppressing the monomial factor, $5a^3$, in the numerator, the greatest common divisor of the two terms may be found as follows:

$$\begin{array}{l|l}
a^3x + 2a^2x^2 + 2ax^3 + x^4 & a^2 + 2ax + x^2 \\
\underline{a^3x + 2a^2x^2 + \quad ax^3} & (ax \\
\qquad\qquad\qquad ax^3 + x^4 & \\
\end{array}$$

$$\begin{array}{r}
a + x)\ a^2 + 2ax + x^2\ (a + x \\
\underline{a^2 + \quad ax \qquad} \\
ax + x^2 \\
\underline{ax + x^2}
\end{array}$$

Dividing both terms of the given fraction by the greatest common divisor, $a + x$, we obtain at once $\dfrac{5a^4 + 5a^3x}{a^2x + ax^2 + x^3}$.

(Art. **125**, pp. 80, 81.)

3. $\dfrac{x^2-y^2}{x+y}=\dfrac{(x+y)(x-y)}{x+y}=x-y.$

(4.)

$$\begin{array}{l} a+x)\ a\,x+2\,x^2\ (x+\dfrac{x^2}{a+x} \\ \qquad\quad \underline{a\,x+\ \ x^2} \\ \qquad\qquad\qquad x^2 \end{array}$$

5. $\dfrac{x^3-y^3}{x-y}=\dfrac{(x-y)(x^2+x\,y+y^2)}{x-y}=x^2+x\,y+y^2.$

6. $\dfrac{12\,x^2-18}{3\,x}=4\,x-\dfrac{18}{3\,x}=4\,x-\dfrac{6}{x}$, Ans.

7. $$\begin{array}{l} 2\,x^2-x+1)\ 4\,x^2-2\,x\quad (2-\dfrac{2}{2\,x^2-x+1} \\ \qquad\qquad\quad \underline{4\,x^2-2\,x+2} \\ \qquad\qquad\qquad\qquad -2 \end{array}$$

8. $\dfrac{a^3-b^3+x^3}{a+x}=\dfrac{a^3+x^3}{a+x}-\dfrac{b^3}{a+x}=\dfrac{(a+x)(a^2-a\,x+x^2)}{a+x}$

$-\dfrac{b^3}{a+x}=a^2-a\,x+x^2-\dfrac{b^3}{a+x}.$

(Art. **126**, pp. 81, 82.)

3. $\dfrac{5\,a\,b^2}{3\,b^3\,c^2}=\dfrac{5\,a}{3\,b\,c^2}=5\times 3^{-1}\,a\,b^{-1}\,c^{-2}.$

4. $\dfrac{3\,x\,y^3}{3\,x^2\,y}=\dfrac{y^2}{x}=x^{-1}\,y^2.$ Ans.

6. $\frac{x^2 y^3}{a^{-3} y^{-2}} = a^3 x^2 y^3 y^2 = a^3 x^2 y^5.$

8. $\frac{a - b}{(a + b)^{-1}} = (a - b)(a + b) = a^2 - b^2.$

9. $\frac{4 a^{-3} b^3 x^{-2}}{2 a^{-2} b^{-2}} = \frac{4 a^2 b^3 b^2}{2 a^3 x^2} = \frac{2 b^5}{a x^2}.$

10. $\frac{x y (a - b)^{-2}}{a + b} = \frac{x y}{(a + b)(a - b)^2} = \frac{x y}{a^3 - a^2 b - a b^2 + b^3}.$

In the above examples, factors are transferred from either term of a fraction to the other by changing the signs of their exponents.

Since the fractional form may be regarded as an expression of division, the required result may also be obtained, in each of the first eight examples of Art. 126, by simply subtracting the exponents found in the denominator from those connected with the same letters or quantities in the numerator. The exponent of any letter or quantity not found in the numerator may be regarded as 0. (Art. 70.)

(Art. **127**, pp. 83, 84.)

4. Ans. $\frac{a b + a^2 x^3}{a}.$

5. $a - \frac{a b - a^2}{2 b} = \frac{2 a b - (a b - a^2)}{2 b} = \frac{2 a b - a b + a^2}{2 b}$

$= \frac{a b + a^2}{2 b}.$

6. $a + 1 - \frac{x - 1}{b} = \frac{a b + b - (x - 1)}{b} = \frac{a b + b - x + 1}{b}.$

7. $2a - 2b + \frac{a-x}{3} = \frac{6a - 6b + a - x}{3} = \frac{7a - 6b - x}{3}.$

8. $1 + 3a - \frac{4x-5}{4x} = \frac{4x + 12ax - (4x - 5)}{4x} = \frac{12ax + 5}{4x}.$

9. $a + b - \frac{a^2 - b^2 - 3}{a - b} = \frac{a^2 - b^2 - (a^2 - b^2 - 3)}{a - b} = \frac{3}{a - b}.$

10. $2 + \frac{x^2 + y^2}{xy} = \frac{2xy + x^2 + y^2}{xy} = \frac{(x+y)^2}{xy}.$

11. $a + b - \frac{a^2 - 2ab + b^2}{a + b} = \frac{a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)}{a + b}$
$= \frac{4ab}{a+b}.$

(Art. 128, page 86.)

4. The least common multiple of the denominators $5y$, $10y^2$, and $2x$, is $10xy^2$. (Art. 105.)

$(10xy^2 \div 5y) \times 4x = 8x^2y; \quad \frac{4x}{5y} = \frac{8x^2y}{10xy^2}.$

$(10xy^2 \div 10y^2) \times 7m = 7mx; \quad \frac{7m}{10y^2} = \frac{7mx}{10xy^2}.$

$(10xy^2 \div 2x) \times n = 5ny^2; \quad \frac{n}{2x} = \frac{5ny^2}{10xy^2}.$

5. The least common multiple of $2a$, $5a^2$, and n is $10a^2n$.

$(10a^2n \div 2a) \times 3x = 15anx; \quad \frac{3x}{2a} = \frac{15anx}{10a^2n}.$

$(10a^2n \div 5a^2) \times 2x = 4nx; \quad \frac{2x}{5a^2} = \frac{4nx}{10a^2n}.$

$(10a^2n \div n) \times m = 10a^2m; \quad \frac{m}{n} = \frac{10a^2m}{10a^2n}.$

Ans. $\frac{15anx}{10a^2n}, \frac{4nx}{10a^2n}, \frac{10a^2m}{10a^2n}.$

Examples 6 and 7 are readily solved by the first part of the rule.

8. The least common multiple of 1, b, and $c + d$ is $b\,(c + d)$, or $b\,c + b\,d$.

$$[(b\,c + b\,d) \div 1] \times a = a\,b\,c + a\,b\,d;\ \frac{a}{1} = \frac{a\,b\,c + a\,b\,d}{b\,c + b\,d}.$$

$$[(b\,c + b\,d) \div b] \times a = \quad a\,c + a\,d;\ \ \frac{a}{b} = \frac{a\,c + a\,d}{b\,c + b\,d}.$$

$$[(bc + bd) \div (c + d)] \times (x - 2) = bx - 2b;\ \frac{x - 2}{c + d} = \frac{b\,x - 2\,b}{b\,c + b\,d}.$$

NOTE. The same results may also be obtained by means of the first part of the rule.

(ART. **129**, p. 87.)

2. Since $(x + y)\,(x - y) = x^2 - y^2$, it is evident that if we multiply both numerator and denominator of the first fraction by $x - y$, and of the second by $x + y$, the denominator of each of the three fractions will be $x^2 - y^2$.

3. Since $(a - x)\,(a^2 + a\,x + x^2) = a^3 - x^3$ (Art. 85), it is evident that the multiplier $a^2 + a\,x + x^2$ should be used for the second fraction, and $a - x$ for the third.

ADDITION.

(ART. **131**, pp. 88, 89.)

3. $\frac{3\,a}{4} + \frac{5\,a}{6} + \frac{a}{3} = \frac{9\,a}{12} + \frac{10\,a}{12} + \frac{4\,a}{12} = \frac{23\,a}{12}.$

4. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{6\,x}{12} + \frac{4\,x}{12} + \frac{3\,x}{12} = \frac{13\,x}{12}.$

5. $\frac{3x}{2a}+\frac{x}{5}=\frac{15x}{10a}+\frac{2ax}{10a}=\frac{15x+2ax}{10a}$.

7. $\frac{4x}{7}+\frac{x-2}{5}=\frac{20x}{35}+\frac{7x-14}{35}=\frac{27x-14}{35}$.

8. $\frac{3x-a}{14}+\frac{4x}{7}=\frac{3x-a}{14}+\frac{8x}{14}=\frac{11x-a}{14}$, Ans.

9. $\frac{x}{x+y}+\frac{y}{x-y}=\frac{x^2-xy}{x^2-y^2}+\frac{xy+y^2}{x^2-y^2}=\frac{x^2+y^2}{x^2-y^2}$.

10. $\frac{1}{1+a}+\frac{1}{1-a}=\frac{1-a}{1-a^2}+\frac{1+a}{1-a^2}=\frac{2}{1-a^2}$.

11. $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{x^2+2xy+y^2}{x^2-y^2}+\frac{x^2-2xy+y^2}{x^2-y^2}$

$$=\frac{2x^2+2y^2}{x^2-y^2}=\frac{2(x^2+y^2)}{x^2-y^2}.$$

12. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ac}=\frac{ac-bc}{abc}+\frac{ab-ac}{abc}+\frac{bc-ab}{abc}$

$$=\frac{0}{abc}=0.$$

13. $\frac{x-a}{x^2-ax+a^2}+\frac{1}{x+a}=\frac{x^2-a^2}{x^3+a^3}+\frac{x^2-ax+a^2}{x^3+a^3}$

$$=\frac{2x^2-ax}{x^3+a^3}.$$

14. $\frac{n}{n-1}+\frac{1-2n}{n^2-n}=\frac{n^2}{n^2-n}+\frac{1-2n}{n^2-n}=\frac{n^2-2n+1}{n^2-n}$

$$=\frac{(n-1)(n-1)}{n(n-1)}=\frac{n-1}{n}.$$

16. $3a+\frac{2x}{5}+a-\frac{8x}{9}=3a+a+\frac{18x}{45}-\frac{40x}{45}=4a-\frac{22x}{45}$.

17. $c+\frac{c+d}{2}+\frac{c-d}{2}=c+\frac{2c}{2}=c+c=2c$.

18. $x - \frac{4a^2}{b} + y + \frac{2ax}{c} = x + y - \frac{4a^2c}{bc} + \frac{2abx}{bc}$

$$= x + y + \frac{2abx - 4a^2c}{bc}.$$

SUBTRACTION.

(Art. 133, pp. 91, 92.)

3. $\frac{12x}{7} - \frac{3x}{5} = \frac{60x}{35} - \frac{21x}{35} = \frac{39x}{35}.$

4. $\frac{a}{3} - \frac{a}{4} = \frac{4a}{12} - \frac{3a}{12} = \frac{a}{12}.$

5. $\frac{3ab}{4} - \frac{4ab}{2} = \frac{3ab}{4} - \frac{8ab}{4} = -\frac{5ab}{4}.$

6. $\frac{x+y}{2} - \frac{x-y}{2} = \frac{2y}{2} = y.$

7. $\frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1}{x^2-1} - \frac{x-1}{x^2-1} = \frac{2}{x^2-1}$, Ans.

8. $\frac{1-2n}{n^2-n} - \frac{n}{n-1} = \frac{1-2n}{n^2-n} - \frac{n^2}{n^2-n} = \frac{1-2n-n^2}{n^2-n}.$

9. $3x - \frac{x}{2b} - \left(x - \frac{x-a}{c}\right) = 3x - x - \frac{cx}{2bc} + \frac{2bx-2ab}{2bc}$

$$= 2x + \frac{2bx - 2ab - cx}{2bc} = \frac{4bcx + 2bx - cx - 2ab}{2bc}.$$

10. $\frac{1+x^2}{1-x^2} - \frac{4x^2}{1-x^4} = \frac{(1+x^2)^2}{1-x^4} - \frac{4x^2}{1-x^4} = \frac{1+2x^2+x^4-4x^2}{1-x^4}$

$$= \frac{1-2x^2+x^4}{1-x^4} = \frac{(1-x^2)(1-x^2)}{(1-x^2)(1+x^2)} = \frac{1-x^2}{1+x^2}.$$

11. $4x + \frac{b}{a} - \left(3x - \frac{c}{d}\right) = 4x - 3x + \frac{bd}{ad} + \frac{ac}{ad}$

$= x + \frac{bd + ac}{ad}.$

12. $7b - \frac{4a+1}{2} - \left(b + \frac{3}{5}\right) = 7b - b - \frac{20a+5}{10} - \frac{6}{10}$

$= 6b - \frac{20a+11}{10}.$

13. $\frac{3x+2}{y} - \frac{7xy - 10y}{y^2} = \frac{3x+2}{y} - \frac{7x - 10}{y}$

$= \frac{3x+2-7x+10}{y} = \frac{12-4x}{y} = \frac{4(3-x)}{y}.$

14. $1 - \frac{x-a}{x+a} = \frac{x+a}{x+a} - \frac{x-a}{x+a} = \frac{x+a-x+a}{x+a} = \frac{2a}{x+a}.$

15. $2x + \frac{3x}{a} - x - \frac{2x-2a}{3c} = 2x - x + \frac{9cx}{3ac} - \frac{2ax-2a^2}{3ac}$

$= x + \frac{9cx - 2ax + 2a^2}{3ac}.$

16. $\frac{xy}{x-y} - \frac{xy}{x+y} = \frac{x^2y + xy^2}{x^2 - y^2} - \frac{x^2y - xy^2}{x^2 - y^2}$

$= \frac{x^2y + xy^2 - x^2y + xy^2}{x^2 - y^2} = \frac{2xy^2}{x^2 - y^2}.$

MULTIPLICATION.

(ART. 135, page 93.)

8. $\frac{b-c}{ab+ac+bc+c^2} \times (a+c) = \frac{(b-c)(a+c)}{(b+c)(a+c)} = \frac{b-c}{b+c}.$

9. $\frac{a+b+c}{9(x-y)(x+y)} \times 3(x+y) = \frac{3(a+b+c)(x+y)}{9(x-y)(x+y)}$

$= \frac{a+b+c}{3(x-y)}.$

(ART. 137, pp. 95, 96.)

6. Ans. $\frac{b\,c\,m^2}{a\,d}$.

11. $\left(a+\frac{x}{y}\right)\times\frac{y}{a}=\frac{a\,y+x}{y}\times\frac{y}{a}=\frac{a\,y+x}{a}$.

A factor occurring in any numerator will cancel the same factor found in any denominator, when the fractions are to be multiplied together.

13. $\frac{4\,x+2}{3}\times\frac{5\,x}{2\,x+1}=\frac{2\,(2\,x+1)}{3}\times\frac{5\,x}{2\,x+1}=\frac{10\,x}{3}$.

14. $\frac{a+b}{b}\times\frac{b^2\,x}{a^2-b^2}=\frac{a+b}{b}\times\frac{b^2\,x}{(a+b)\,(a-b)}=\frac{b\,x}{a-b}$.

16. $\frac{a^2+b^2}{a^2-b^2}\times\frac{a-b}{a+b}=\frac{a^2+b^2}{(a+b)\,(a-b)}\times\frac{a-b}{a+b}=\frac{a^2+b^2}{(a+b)^2}$
$=\frac{a^2+b^2}{a^2+2\,a\,b+b^2}$.

17. $\frac{a-b}{a}\times\frac{a+b}{b}\times\frac{a^2}{a^2-b^2}=\frac{a-b}{a}\times\frac{a+b}{b}$
$\times\frac{a^2}{(a+b)\,(a-b)}=\frac{a}{b}$.

18. $\left(a-\frac{b^2}{a}\right)\left(\frac{a}{b}+\frac{b}{a}\right)=\frac{a^2-b^2}{a}\times\frac{a^2+b^2}{a\,b}=\frac{a^4-b^4}{a^2\,b}$.

19. $\left(\frac{x}{x+y}\right)\left(\frac{1}{x-y}\right)\left(\frac{x^2-y^2}{x}\right)=\frac{x}{x+y}\times\frac{1}{x-y}$
$\times\frac{(x+y)\,(x-y)}{x}=1$.

20. $\left(a+\frac{b}{y}\right)\times\left(x+\frac{m}{n}\right)=\frac{a\,y+b}{y}\times\frac{n\,x+m}{n}$
$=\frac{a\,n\,x\,y+a\,m\,y+b\,n\,x+b\,m}{n\,y}$

DIVISION.

(ART. 139, pp. 97, 98.)

6. $\frac{x+xy}{1+y} \div x = \frac{x(1+y)}{1+y} \div x = \frac{1+y}{1+y} = 1.$

7. $\frac{abc+b^2c}{x} \div (a+b) = \frac{bc(a+b)}{x} \div (a+b) = \frac{bc}{x}.$

8. $\frac{a^2d+abd}{c} \div ad = \frac{ad(a+b)}{c} \div ad = \frac{a+b}{c}$, Ans.

9. $\frac{b-c}{b+c} \div (a+c) = \frac{b-c}{(b+c)(a+c)} = \frac{b-c}{ab+ac+bc+c^2}.$

(ART. 140, pp. 99 - 101.)

3. $\frac{3ay}{4m} \div \frac{2m}{y} = \frac{3ay}{4m} \times \frac{y}{2m} = \frac{3ay^2}{8m^2}.$

4. $\frac{a}{2} \div \frac{3m}{4a} = \frac{a}{2} \times \frac{4a}{3m} = \frac{2a^2}{3m}.$

5. $\frac{4a^2}{9m} \div \frac{2a}{3} = \frac{4a^2}{9m} \times \frac{3}{2a} = \frac{2a}{3m}.$

6. $\frac{8a}{9m} \div \frac{a}{n} = \frac{8a}{9m} \times \frac{n}{a} = \frac{8n}{9m}$, Ans.

7. $\frac{ab}{1} \div \frac{3d}{2ab} = \frac{ab}{1} \times \frac{2ab}{3d} = \frac{2a^2b^2}{3d}.$

8. $\frac{ay}{1} \div \frac{a-b}{x} = \frac{ay}{1} \times \frac{x}{a-b} = \frac{axy}{a-b}.$

9. $\frac{a}{1-a} \div \frac{a}{4} = \frac{a}{1-a} \times \frac{4}{a} = \frac{4}{1-a}.$

10. $\frac{a+b}{2} \div \frac{a-b}{2} = \frac{a+b}{2} \times \frac{2}{a-b} = \frac{a+b}{a-b}.$

11. $\left(x+\frac{a}{y}\right) \div \frac{y}{a} = \frac{xy+a}{y} \times \frac{a}{y} = \frac{axy+a^2}{y^2}.$

12. $\frac{3a+b}{6} \div \frac{ab+m}{7a} = \frac{3a+b}{6} \times \frac{7a}{ab+m} = \frac{21a^2+7ab}{6ab+6m}.$

13. $\frac{a^2m+bmy}{an} \div \left(a+\frac{by}{a}\right) = \frac{a^2m+bmy}{an} \div \frac{a^2+by}{a}$

$= \frac{m(a^2+by)}{an} \times \frac{a}{a^2+by} = \frac{m}{n}.$

14. $\frac{a+1}{a} \div \frac{a^2-1}{a} = \frac{a+1}{a} \times \frac{a}{a^2-1} = \frac{a+1}{a}$

$\times \frac{a}{(a+1)(a-1)} = \frac{1}{a-1}.$

15. $\frac{6x^2-2x}{4-x^2} \div \frac{x^2}{2+x} = \frac{x(6x-2)}{(2-x)(2+x)} \times \frac{2+x}{x^2}$

$= \frac{6x-2}{x(2-x)} = \frac{6x-2}{2x-x^2}.$

16. $\frac{2b^2}{a^3+b^3} \div \frac{b}{a+b} = \frac{2b^2}{(a+b)(a^2-ab+b^2)} \times \frac{a+b}{b}$

$= \frac{2b}{a^2-ab+b^2}.$

17. $\frac{x^2-y^2}{x+2y} \div \frac{x-y}{3x+6y} = \frac{(x+y)(x-y)}{x+2y} \times \frac{3(x+2y)}{x-y}$

$= 3(x+y).$

18. $(1+x) \div \frac{1+x}{x} = \frac{1+x}{1} \times \frac{x}{1+x} = x.$

19. $12 \div \left(\frac{(a+x)^2}{x} - a\right) = \frac{12}{1} \div \frac{a^2 + a\,x + x^2}{x}$

$$= \frac{12}{1} \times \frac{x}{a^2 + a\,x + x^2} = \frac{12\,x}{a^2 + a\,x + x^2}.$$

(Art. **142**, pp. 102, 103.)

2. $\frac{x}{y} \div \frac{m}{n} = \frac{x}{y} \times \frac{n}{m} = \frac{n\,x}{m\,y}.$

3. $\frac{a+b}{x} \div \frac{y-a}{b} = \frac{a+b}{x} \times \frac{b}{y-a} = \frac{a\,b + b^2}{x\,y - a\,x}.$

4. Multiply both numerator and denominator of the complex fraction by b, according to the second part of the rule, and the answer is obtained at once.

5. $\frac{5-c}{x} \div \frac{7-y}{a} = \frac{5-c}{x} \times \frac{a}{7-y} = \frac{5\,a - a\,c}{7\,x - x\,y}.$ Or, multiply both terms of the complex fraction by $a\,x$.

6. Such a fraction as $\frac{y - m\,n^{-1}}{x + a\,d^{-1}}$ is complex in meaning, though not in form, being equivalent to $\frac{y - \frac{m}{n}}{x + \frac{a}{d}}$. It is evident from the rule that the denominators n and d may be removed by multiplying both terms of the complex fraction by $d\,n$; and, according to the note, the negative exponents of the original form are removed by the use of the same multiplier, thus:

$$\frac{(y - m\,n^{-1}) \times d\,n}{(x + a\,d^{-1}) \times d\,n} = \frac{d\,n\,y - d\,m}{d\,n\,x + a\,n}.$$

7. $\frac{b}{x+a^{-1}}=\frac{b\times a}{(x+a^{-1})\,a}=\frac{a\,b}{a\,x+a^0}=\frac{a\,b}{a\,x+1}.$ (Art. 70.)

8. $$\frac{\frac{1}{b-x}}{b+x}=\frac{\frac{1}{b-x}\times(b-x)}{(b+x)\times(b-x)}=\frac{1}{b^2-x^2}.$$

9. $$\frac{\frac{2\,x^2-b}{5}}{\frac{a-b}{4}}=\frac{\frac{2\,x^2-b}{5}\times 20}{\frac{a-b}{4}\times 20}=\frac{4\,(2\,x^2-b)}{5\,(a-b)}=\frac{8\,x^2-4\,b}{5\,a\;-5\,b}.$$

10. The operation for reducing the given complex fraction to a simple one, according to the rule, is as follows:

$$\frac{x^4-\frac{1}{x^4}}{x+\frac{1}{x}}=\frac{\left(x^4-\frac{1}{x^4}\right)\times x^4}{\left(x+\frac{1}{x}\right)\times x^4}=\frac{x^8-1}{x^5+x^3}.$$

The last expression may be reduced thus (Art. 86):

$$\frac{x^8-1}{x^5+x^3}=\frac{(x^2+1)\,(x^6-x^4+x^2-1)}{x^3\,(x^2+1)}=\frac{x^6-x^4+x^2-1}{x^3}$$

$$=x^3-x+\frac{1}{x}-\frac{1}{x^3}.$$

Again, $\dfrac{x^4-\frac{1}{x^4}}{x+\frac{1}{x}}=\dfrac{x^4-x^{-4}}{x+x^{-1}}=x^3-x+x^{-1}-x^{-3}.$ In performing the division, negative exponents must be subtracted or added whenever positive ones would be; but these operations must be performed in accordance with the principles already laid down in the case of other negative quantities, thus:

$$
\begin{array}{l|l}
x^4 - x^{-4} & x + x^{-1} \\ \cline{2-2}
x^4 + x^2 & x^3 - x + x^{-1} - x^{-3} \\ \hline
\quad - x^2 - x^{-4} & \\
\quad - x^2 - x^0 & \\ \cline{1-1}
\qquad x^0 - x^{-4} & \\
\qquad x^0 + x^{-2} & \\ \cline{1-1}
\qquad\quad - x^{-2} - x^{-4} & \\
\qquad\quad - x^{-2} - x^{-4} & \\ \cline{1-1}
\end{array}
\qquad
\begin{array}{l|l}
x^4 - \frac{1}{x^4} & x + \frac{1}{x} \\ \cline{2-2}
x^4 + x^2 & x^3 - x + \frac{1}{x} - \frac{1}{x^3} \\ \cline{1-1}
\quad - x^2 - \frac{1}{x^4} & \\
\quad - x^2 - 1 & \\ \cline{1-1}
\qquad 1 - \frac{1}{x^4} & \\
\qquad 1 + \frac{1}{x^2} & \\ \cline{1-1}
\qquad\quad - \frac{1}{x^2} - \frac{1}{x^4} & \\
\qquad\quad - \frac{1}{x^2} - \frac{1}{x^4} & \\ \cline{1-1}
\end{array}
$$

SIMPLE EQUATIONS.

TRANSFORMATION OF EQUATIONS.

(ART. **152**, pp. 106, 107.)

5. Ans. $5x - 2x = 24 - 3$.

NOTE. The student must always bear in mind the fact that transposition is simply adding to, or subtracting from, both members of an equation a quantity equal in numerical value to the one which is to be transposed, according to the first and second axioms. It may be well for the teacher to require this explanation of each transposition, until the idea is firmly fixed in the student's mind. Thus, in the fifth example, we subtract 3 and $2x$ from both members of the equation, according to Ax. 2.

(ART. **153**, pp. 108, 109.)

2. Multiply by a.
3. Multiply by 4.
4. Multiply by 6.
5. Multiply by 10.
6. Multiply by x.

7. Given $m + x^{-1} = n - p x^{-1}$
Multiplying by x, $m x + x^0 = n x - p x^0$
Or, $m x + 1 = n x - p$

NOTE. It will be seen that this example is precisely analogous to the sixth; but the equation takes a different form, merely because negative exponents are used instead of denominators.

The student should become accustomed to the use of negative exponents as early as possible. He will find that they may often be convemiently substituted for fractional forms; in fact, they constitute the natural algebraic expression of unperformed division, just as a negative coefficient expresses an unperformed subtraction. The ability to comprehend and employ them will also be found essential in the higher departments of mathematical analysis.

8. Multiply by 2. Ans. $2a - 1 + 2b = 2c + d - 2x$.

9. Given $x - \frac{4x + 8}{6} = 8$

Multiplying by 6, $6x - (4x + 8) = 48$
Hence (Note 2), $6x - 4x - 8 = 48$

12. When the denominators are prime to each other, it is convenient to multiply each numerator by all the denominators except its own. In this instance, therefore, we may thus multiply both members of the equation by 55.

NOTE. Here, as in transposition, the student must be reminded that he can perform no operation upon one member of an equation without also performing precisely the same operation upon the other member. The process of clearing of fractions is based entirely upon the third axiom. Multiplying each numerator by all the denominators except its own is simply a convenient practical expedient for multiplying both members by the common multiple of all the denominators.

13. Multiply by 6, and apply Note 2.

14. Multiplying both members of this equation by bc removes both denominators and the negative exponent

15. Multiply both members of the equation by 8, and develop the expression $3(44 - x + 12)$ for the second member.

16. Multiplying both members of the equation by 9 removes all the denominators, and gives

$$6x - 60 = 4(30 - x).$$

Performing the multiplication indicated in the second member, we obtain

$$6x - 60 = 120 - 4x.$$

NOTE. In multiplying a fractional term by the least common multiple of the denominators, we may first divide this multiple by the denominator of the term, and then multiply the numerator by the quotient, as in reducing fractions to a common denominator. (Art. 128.)

17 Given $\quad \dfrac{x-2}{x+2} + \dfrac{x+2}{x-2} = 14.$

Multiplying both members by $(x+2)(x-2)$, that is each numerator by the denominator of the other fraction, and 14 by both denominators,

$$(x-2)^2 + (x+2)^2 = 14(x+2)(x-2)$$

Or, $\quad x^2 - 4x + 4 + x^2 + 4x + 4 = 14(x^2 - 4)$

SOLUTION OF SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

(ART. 159, pp. 112 - 114.)

5. Given $\quad 5x + 43 - 5 = 100 - 27$

Transposing the known terms to the second member,

$$5x = 100 - 27 - 43 + 5$$

Uniting the known terms, $5x = 85$

Dividing both members by 5, the coefficient of x,

$$x = 7$$

6. Given $7x + 7 + 1 = 96 - 11$
Transposing known terms, $7x = 96 - 11 - 7 - 1$
Uniting terms, $7x = 77$
Dividing by 7, $x = 11$

7. Given $15x + 8 - 9 = 212 + 87$
Transposing terms, $15x = 212 + 87 - 8 + 9$
Uniting terms, $15x = 300$
Dividing by 15, $x = 20$

8. Given $9x + 9 = x - 71$
Transposing terms, $9x - x = -71 - 9$
Uniting terms, $8x = -80$
Dividing by 8, $x = -10$

9. Given $4x - 15 = 2x + 13$
Transposing terms, $4x - 2x = 13 + 15$
Uniting terms, $2x = 28$
Dividing by 2, $x = 14$

10. Given $4(x - 12) = 2(12 - x)$
Expanding, $4x - 48 = 24 - 2x$
Transposing terms, $4x + 2x = 24 + 48$
Uniting terms, $6x = 72$
Dividing by 6, $x = 12$

11. Given $9(x + 1) = 12(x - 2)$
Expanding, $9x + 9 = 12x - 24$
Transposing terms, $9x - 12x = -24 - 9$
Uniting terms, $-3x = -33$
Dividing by -3, $x = 11$

12. Given $3(x - 3) + 2x = 3(40 - x - 19)$
Expanding, $3x - 9 + 2x = 120 - 3x - 57$
Transposing, $3x + 2x + 3x = 120 - 57 + 9$
Uniting, $8x = 72$
Dividing by 8, $x = 9$

13. Given $3\,(2\,x + 3\,x) - 15 = 72 - 4\,(x - 2)$
Expanding, $6\,x + 9\,x - 15 = 72 - 4\,x + 8$
Transposing, $6\,x + 9\,x + 4\,x = 72 + 8 + 15$
Uniting, $19\,x = 95$
Dividing by 19, $x = 5$

14. Given $2\,(x - 6) + 3\,(2\,x + 5) = 3\,(3\,x - 2) - 1$
Expanding, $2x - 12 + 6x + 15 = 9\,x - 6 - 1$
Transposing, $2\,x + 6\,x - 9\,x = -6 - 1 + 12 - 15$
Uniting, $-x = -10$
Changing signs, $x = 10$ (Art. 152, Note.)

15. Given $x - \frac{x}{2} - \frac{x}{6} = 30$
Clearing of fractions by multiplying by 6,
$$6\,x - 3\,x - x = 180$$
Uniting terms, $2\,x = 180$
Dividing by 2, $x = 90$

16. FIRST SOLUTION.

Given $1 - 8\,x^{-1} = \frac{1}{5} + 8\,x^{-1}$
Multiplying by $5\,x$, to remove denominators and negative exponents, $5\,x - 40 = x + 40$
Transposing terms, $5\,x - x = 40 + 40$
Uniting terms, $4\,x = 80$
Dividing by 4, $x = 20$

SECOND SOLUTION.

Given $1 - 8\,x^{-1} = \frac{1}{5} + 8\,x^{-1}$
Multiplying by 5, $5 - 40\,x^{-1} = 1 + 40\,x^{-1}$
Transposing, $-40\,x^{-1} - 40\,x^{-1} = 1 - 5$
Uniting, $-80\,x^{-1} = -4$
Dividing by -80, $x^{-1} = \frac{4}{80} = \frac{1}{20}$
Taking the reciprocals, $x = 20$

17. Given $\frac{2}{3}x + 12 = \frac{4}{5}x + 6$

Multiplying by 15, $10x + 180 = 12x + 90$
Transposing, $10x - 12x = 90 - 180$
Uniting, $-2x = -90$
Dividing by -2, $x = 45$

18. Given $\frac{x}{3} + \frac{3x}{5} + \frac{4x}{7} = 158$

Multiplying by 105, $35x + 63x + 60x = 16590$
Uniting, $158x = 16590$
Dividing by 158, $x = 105$

Note. In clearing of fractions, the multiplication of the second member need only be expressed, thus: 158×105. The factor 158 will then be removed by the next division, and leave the other factor, 105, as the value of x.

19. Given $\frac{x}{2} + \frac{x}{4} + \frac{x}{8} = 28$

Multiplying by 8, $4x + 2x + x = 224$
Uniting, $7x = 224$
Dividing by 7, $x = 32$

21. Given $\frac{ax}{n} = d$

Multiplying by n, $ax = nd$

Dividing by a, $x = \frac{nd}{a}$

22. Given $\frac{ax}{2} + \frac{bx}{3} = c$

Multiplying by 6, $3ax + 2bx = 6c$
Factoring, $(3a + 2b)x = 6c$

Dividing by $3a + 2b$, $x = \frac{6c}{3a + 2b}$

23. Given $\frac{1}{a} + \frac{b}{x} = c$

Multiplying by $a\,x$, $x + a\,b = a\,c\,x$

Transposing, $x - a\,c\,x = -a\,b$

Changing signs, $a\,c\,x - x = a\,b$ (Art. 152, Note.)

Factoring, $(a\,c - 1)\,x = a\,b$

Dividing by $a\,c - 1$, $x = \frac{a\,b}{a\,c - 1}$

24. Given $x + n\,x = a$

Factoring, $(1 + n)\,x = a$

Dividing by $1 + n$, $x = \frac{a}{1 + n}$

25. FIRST SOLUTION.

Given $a - c\,x^{-1} = d\,x^{-1} - b$

Multiplying by x, $a\,x - c = d - b\,x$

Transposing, $a\,x + b\,x = c + d$

Factoring, $(a + b)\,x = c + d$

Dividing by $a + b$, $x = \frac{c + d}{a + b}$

SECOND SOLUTION.

Given $a - c\,x^{-1} = d\,x^{-1} - b$

Transposing, $-c\,x^{-1} - d\,x^{-1} = -a - b$

Changing signs, $c\,x^{-1} + d\,x^{-1} = a + b$

Factoring, $(c + d)\,x^{-1} = a + b$

Dividing by $c + d$, $x^{-1} = \frac{a + b}{c + d}$

Taking reciprocals, $x = \frac{c + d}{a + b}$

26. Given $\frac{b\,x}{a} - \frac{d}{c} = \frac{a}{b} - \frac{c\,x}{d}$

Mult. by $a\,b\,c\,d$, $b^2\,c\,d\,x - a\,b\,d^2 = a^2\,c\,d - a\,b\,c^2\,x$

Transposing, $a\,b\,c^2\,x + b^2\,c\,d\,x = a^2\,c\,d + a\,b\,d^2$

Factoring, $b\,c\,x\,(a\,c + b\,d) = a\,d\,(a\,c + b\,d)$
Dividing by $a\,c + b\,d$, $b\,c\,x = a\,d$
Dividing by $b\,c$, $x = \frac{a\,d}{b\,c}$

Note. If the members of the equation had not been factored and divided by $a\,c + b\,d$, we should have obtained

$$x = \frac{a^2\,c\,d + a\,b\,d^2}{a\,b\,c^2 + b^2\,c\,d} = \frac{a\,d\,(a\,c + b\,d)}{b\,c\,(a\,c + b\,d)} = \frac{a\,d}{b\,c}.$$

27. Given $\frac{2\,x + 1}{3} = x - \frac{x + 3}{4}$
Multiplying by 12, $8\,x + 4 = 12\,x - 3\,x - 9$
Transposing, $8\,x - 12\,x + 3\,x = -9 - 4$
Uniting, $-x = -13$
Changing signs, $x = 13$

29. Given $5 - \frac{x + 4}{11} = x - 3$
Multiplying by 11, $55 - x - 4 = 11\,x - 33$
Transposing, $-x - 11\,x = -33 + 4 - 55$
Uniting, $-12\,x = -84$
Dividing by -12, $x = 7$

30. Given $2\,x + \frac{24}{4} = \frac{3\,x}{4} - 4$
Multiplying by 4, $8\,x + 24 = 3\,x - 16$
Transposing, $8\,x - 3\,x = -16 - 24$
Uniting, $5\,x = -40$
Dividing by 5, $x = -8$

31. Given $x + \frac{x + 8}{4} - \frac{x - 6}{3} = x - 2$
Dropping x from both members, $\frac{x + 8}{4} - \frac{x - 6}{3} = -2$
Multiplying by 12, $3\,x + 24 - 4\,x + 24 = -24$
Transposing, $3\,x - 4\,x = -24 - 24 - 24$
Uniting, $-x = -72$
Changing signs, $x = 72$

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

(ART. 167, pp. 117–131.)

2. Let $x =$ the smaller number,
and $x + 16 =$ the larger number.

Their sum, $x + x + 16 = 40$
Transp. and uniting, $2x = 24$
Dividing by 2, $x = 12$, the smaller number.
Then, $x + 16 = 28$, the larger number.

VERIFICATION.

$$12 + 28 = 40$$
$$28 - 12 = 16$$

3. Let $x =$ no. votes for unsuccessful candidate,
and $x + 120 =$ no. votes for successful candidate.

Their sum, $x + x + 120 = 1296$
Transp. and uniting, $2x = 1176$
Dividing by 2, $x = 588$, no. votes for one.
Then, $x + 120 = 708$, no. votes for the other.

VERIFICATION.

$$588 + 708 = 1296$$
$$1296 - 708 = 120$$

4. Let $x =$ smaller number,
and $x + 13 =$ larger number.

Their sum and 17, $x + x + 13 + 17 = 62$
Transposing and uniting, $2x = 32$
Dividing by 2, $x = 16$, one number.
Then, $x + 13 = 29$, other number.

5. Let $x =$ sum due A,
$2x =$ sum due B,
and $3x =$ sum due C.

The whole sum, $6x = 3000$
Dividing by 6, $x = 500$, sum due A.
Then, $2x = 1000$, sum due B.
Also, $3x = 1500$, sum due C.

6. Let $x =$ no. of children,
$2x =$ no. of women,
and $4x =$ no. of men.

The whole number, $7x = 266$
Dividing by 7, $x = 38$, no. children.
Then $2x = 76$, no. women.
Also, $4x = 152$, no. men.

8. Let $x =$ number of days required.
Then $36x =$ distance traveled by one,
and $30x =$ dist. traveled by the other.

Their sum, $36x + 30x = 396$
Uniting terms, $66x = 396$
Dividing by 66, $x = 6$, no. days.
Then, $36x = 216$, no. miles one trav.
Also, $30x = 180$, no. miles other trav.

9. Let $x =$ no. hours horseman rides,
and $x + 10 =$ no. hours first person trav.
Then $9x =$ distance horseman rides,
and $4(x + 10) =$ dist. first person travels.

Hence, $9x = 4x + 40$
Transp. and uniting, $5x = 40$
Dividing by 5, $x = 8$, no. hours horseman rides.

10. Let $x =$ price of the house,
and $850 - x =$ price of the garden.

Then, $5x = 12(850 - x)$
Or, $5x = 10200 - 12x$
Transp. and uniting, $17x = 10200$
Dividing by 17, $x = 600$, price of house.
Then, $850 - x = 250$, price of garden.

11. Let $x =$ value of a sheep.
Then $72x + 35 =$ value of A's share,
and $92x - 35 =$ value of B's share.

Hence, $92x - 35 = 72x + 35$
Transp. and uniting, $20x = 70$
Dividing by 20, $x = 3\frac{1}{2}$, value of a sheep.

13. Let $x =$ James's age,
and $\frac{8x}{5} =$ John's age.

The sum, $x + \frac{8x}{5} = 39$
Clear. of fractions, $5x + 8x = 195$
Uniting terms, $13x = 195$
Whence, $x = 15$, James's age.
Then, $\frac{8x}{5} = 24$, John's age.

14. Let $x =$ C's share,

$\frac{2x}{3} =$ A's share,

and $\frac{3x}{4} =$ B's share.

Their sum, $x + \frac{2x}{3} + \frac{3x}{4} = 145$

Clear. of fract., $12x + 8x + 9x = 1740$

Uniting terms, $29x = 1740$

Whence, $x = 60$, C's share.

Then, $\frac{2x}{3} = 40$, A's share.

Also, $\frac{3x}{4} = 45$, B's share.

15 Let $x =$ his age.

Then, $x + \frac{x}{2} + \frac{x}{3} = 44$

Clear. of fract., $6x + 3x + 2x = 264$

Or, $11x = 264$

Whence, $x = 24$, his age.

17. Let $6x =$ length of the pole.

Then $\frac{6x}{2} = 3x =$ half of its length,

and $\frac{6x}{3} = 2x =$ one third of its length.

Hence, $3x + 2x + 4 = 6x$

Or, $-x = -4$

Whence, $x = 4$

Then, $6x = 24$, length of the pole.

18. Let $35x =$ whole journey,

$14x = \frac{2}{5}$ of journey,

and $15x = \frac{3}{7}$ of journey.

Then, $14x + 15x + 30 = 35x$

Or, $-6x = -30$

Whence, $x = 5$

Then, $35x = 175$, the whole journey.

19. Let $6x =$ capacity of the cask,

$2x =$ amount of oil,

and $x = \frac{1}{2}$ of the oil.

Then, $2x - 21 = x$

Or, $x = 21$

Hence, $6x = 126$, capacity of cask.

20. Let $x =$ age of youngest,

$x + 2 =$ age of next,

and $x + 4 =$ age of oldest.

The sum, $3x + 6 = 24$

Transp. and uniting, $3x = 18$

Whence, $x = 6$, age of youngest.

Then, $x + 2 = 8$, age of next.

Also, $x + 4 = 10$, age of oldest.

22. Let $x =$ no. lbs. of each.

Then, $ax + bx + cx = d$

Whence, $x = \frac{d}{a + b + c}$, no. lbs. of each.

23. Let $x =$ my age.

Then, $2x + b = a$

Transposing, $2x = a - b$

Whence, $x = \frac{a - b}{2}$, my age.

24. Let $x =$ no. votes for successful candidate,

and $x - b =$ no. votes for other candidates.

The sum, $x + x - b = a$

Or, $2x = a + b$

Whence, $x = \frac{a + b}{2}$, no. votes for suc. cand.

25. Let $x =$ value of horse,

and $\frac{3x}{2} =$ value of carriage.

Their sum, $x + \frac{3x}{2} = c$

Clear. of fractions, $2x + 3x = 2c$

Or, $5x = 2c$

Whence, $x = \frac{2c}{5}$, value of horse.

Then, $\frac{3x}{2} = \frac{3c}{5}$, val. of carriage

26. Let $x =$ no. days second travels,

and $x + n =$ no. days first travels.

Then $bx =$ distance second travels,

and $a(x + n) =$ distance first travels.

Therefore, $bx = a(x + n)$

Or, $bx = ax + na$

Transposing, $bx - ax = na$

Whence, $x = \frac{na}{b - a}$, no. days required.

27. Let $x =$ the number.

Then, $\frac{7x + 3}{2} - 4 = 15$

Clear. of fractions, $7x + 3 - 8 = 30$

Transposing and uniting, $7x = 35$

Whence, $x = 5$, the number.

28. Let $x =$ length of rod.

Then, $x - \frac{x}{5} - \left(\frac{x}{6} + 3\right) = 16$

Mult. by 30, $30x - 6x - 5x - 90 = 480$

Transp. and uniting, $19x = 570$

Whence, $x = 30$, length rod

30. Let $x =$ price of first house.

Then $x + \frac{x}{2} = \frac{3x}{2} =$ price of second house,

$\frac{3x}{2} + \frac{3x}{4} = \frac{9x}{4} =$ price of third house,

and $x + \frac{9x}{4} = \frac{13x}{4} =$ price of fourth house.

The sum, $x + \frac{3x}{2} + \frac{9x}{4} + \frac{13x}{4} = 8000$

Clearing of fractions,

$4x + 6x + 9x + 13x = 32000$

Uniting, $32x = 32000$

Whence, $x = 1000$, price 1st house.

Then, $\frac{3x}{2} = 1500$, price 2d house.

Also, $\frac{9x}{4} = 2250$, price 3d house.

And, $\frac{13x}{4} = 3250$, price 4th house.

NOTE. Fractions may be avoided by using $4x$ for the price of the first house, and, consequently, $6x$, $9x$, and $13x$ for the prices of the others. The value of x will then be found to be 250.

31. Let $x =$ son's age,

and $3x =$ father's age.

Then, $3x - 5 = 4(x - 5)$

Or, $3x - 5 = 4x - 20$

Transp. and uniting, $-x = -15$

Changing signs, $x = 15$, son's age.

Then, $3x = 45$, father's age.

32. Let $x =$ time of first,

and $10 - x =$ time of second.

Then $14x =$ am't deliv. by first,

and $9(10 - x) =$ am't deliv. by second.

The sum, $14x + 90 - 9x = 120$

6

Transp. and uniting, $5x = 30$

Whence, $x = 6$, time of first.

Then, $10 - x = 4$, time of second.

34. The answer to this problem may be obtained either by substitution in the general formula found by Prob. 33, by a statement similar to that of Problem 33, or as follows:

Let $x =$ the time required.

Then $\frac{1}{x} =$ what both can do in one day.

Also $\frac{1}{3} =$ what A can do in one day,

and $\frac{1}{7} =$ what B can do in one day.

Then, $\frac{1}{3} + \frac{1}{7} = \frac{1}{x}$

Clear. of fract., $7x + 3x = 21$

Or, $10x = 21$

Whence, $x = 2\frac{1}{10}$, time required.

35. The fractional part of the whole work performed by A, B, and C, respectively, in one day will be represented by $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$.

Let $x =$ the time required.

Then $\frac{1}{x} =$ what all can do in one day.

Hence, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{x}$

Clear. of fractions, $bcx + acx + abx = abc$

Whence, $x = \frac{abc}{bc + ac + ab}$

Or, $x = \frac{abc}{ab + ac + bc}$, time req.

NOTE. Since each would work x days, the parts performed by each would be $\frac{x}{a}$, $\frac{x}{b}$, and $\frac{x}{c}$; and as they would perform the whole work in that time, the equation might be stated thus:

$$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1.$$

36. Substituting 2 for a, 3 for b, and 4 for c, in the formula just obtained, we have

$$x=\frac{a\,b\,c}{a\,b+a\,c+b\,c}=\frac{24}{6+8+12}=\frac{24}{26}=\frac{12}{13}$$

$$\tfrac{12}{13} \text{ of } 60 \text{ m.} = 55\tfrac{5}{13} \text{ m.} = 55 \text{ m. } 23\tfrac{1}{13} \text{ s.}$$

This problem may also be solved independently of the preceding formula, thus:

Let $\quad x=$ the time required.

Then, $\quad \frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{1}{x}$

Clearing of fractions, $6\,x+4\,x+3\,x=12$

Or, $\quad 13\,x=12$

Whence, $\quad x=\frac{12}{13}$, the time required.

$$\tfrac{12}{13} \text{ h.} = 55 \text{ m. } 23\tfrac{1}{13} \text{ sec.}$$

38. Let $\quad x=$ no. bushels of rye,

and $\quad x+50=$ no. bushels of the mixture.

Then $\quad \frac{9\,x}{2}=$ value of x bush. rye at $4\frac{1}{2}$ s.,

and $\quad 5\,(x+50)=$ value of the mixture.

Also $50\times 6=300=$ value of the wheat.

Then, $\quad 5\,x+250=\frac{9\,x}{2}+300$

Clearing of fractions, $\quad 10\,x+500=9\,x+600$

Transp. and uniting, $\quad x=100$, no. bush. rye

39. Let $\quad x=$ no. gals. water,

and $\quad x+40=$ no. gals. mixture.

Then, $\quad (x+40)\times 4.50=40\times 7$

Or, $\frac{9x + 360}{2} = 280$

Clearing of fractions, $9x + 360 = 560$

Transp. and uniting, $9x = 200$

Whence, $x = 22\frac{2}{9}$, no. gals. water

41. Let $3x =$ his money.

Then $2x =$ amount at 6 per cent

and $x =$ amount at 8 per cent

$$2x \times \frac{6}{100} = \frac{12x}{100} = \text{income at 6 per cent}$$

$$x \times \frac{8}{100} = \frac{8x}{100} = \text{income at 8 per cent}$$

Then, $\frac{12x}{100} + \frac{8x}{100} = 1200$

Clearing of fractions, $12x + 8x = 120000$

Or, $20x = 120000$

Whence, $x = 6000$, am't at 8 p. c

Then, $2x = 12000$, am't at 6 p.

42. Let $x =$ rent last year,

and $x + \frac{8x}{100} =$ rent this year.

Then, $x + \frac{8x}{100} = 1890$

Clearing of fractions, $100x + 8x = 189000$

Or, $108x = 189000$

Whence, $x = 1750$, rent last year.

43. Let $x =$ the original capital.

Then $x + \frac{40x}{100} - 3000 = \frac{7x}{5} - 3000 =$ capital at the commencement of the second year,

and $\frac{7x}{5} - 3000 + \frac{40}{100}\left(\frac{7x}{5} - 3000\right) - 3000$ $= \frac{7}{5}\left(\frac{7x}{5} - 3000\right) - 3000 =$ capital at the end of the ond year.

Hence, $\frac{7}{5}\left(\frac{7x}{5} - 3000\right) - 3000 = x + \frac{60x}{100}$

Or, $\frac{49x}{25} - \frac{21000}{5} - 3000 = \frac{8x}{5}$

Clear. of fractions, $49x - 105000 - 75000 = 40x$

Transp. and uniting, $9x = 180000$

Whence, $x = 20000$, the capital.

NOTE. The letter a may be used in place of 3000, and we shall obtain $\frac{7x}{5} - a$ and $\frac{7}{5}\left(\frac{7x}{5} - a\right) - a$ as expressions for the capital at the times above specified. The equation becomes

$$\frac{49x}{25} - \frac{7a}{5} - a = \frac{8x}{5},$$

which gives $x = \frac{20a}{8} = 20 \times 1000.$

45. Let $x =$ the amount.

Then, $\frac{3x}{7} + \frac{x}{8} = 930$

Clearing of fractions, $24x + 7x = 930 \times 56$

Or, $31x = 930 \times 56$

Whence, $x = 30 \times 56$

Or, $x = 1680$, the am't req.

47. Let $x =$ the greater part,

and $34 - x =$ the less part.

Then, $x - 18 : 18 - (34 - x) :: 2 : 3$

Or, $x - 18 : x - 16 :: 2 : 3$

Converting the proportion to an equation (Art. 166),

$$3(x - 18) = 2(x - 16)$$

Or, $3x - 54 = 2x - 32$

Transposing and uniting, $x = 22$, the greater part

Then, $34 - x = 12$, the less part.

48 FIRST SOLUTION.

Let $x =$ no. dollar pieces,

and $264 - x =$ no. eagles.

Then, $x : 264 - x :: 9 : 2$

Mult. ext. and means, $2x = 2376 - 9x$
Transp. and uniting, $11x = 2376$
Whence, $x = 216$, no. dollar pieces.
Then, $264 - x = 48$, no. eagles.

SECOND SOLUTION.

Let $9x =$ the number of dollar pieces,
and $2x =$ the number of eagles.

Then, $9x + 2x = 264$
Or, $11x = 264$
Whence, $x = 24$
Then, $2x = 48$, no. eagles.
Also, $9x = 216$, no. dollar pieces.

49 Let $3x =$ age of one,
and $4x =$ age of the other.

Then, $3x - 5 : 4x - 5 :: 2 : 3$
Mult. extremes and means, $9x - 15 = 8x - 10$
Transposing and uniting, $x = 5$
Whence, $3x = 15$, age of one.
Also, $4x = 20$, age of other.

50. FIRST SOLUTION.

Let $x =$ the price per yard.
Then $\frac{5}{x} =$ length of first piece,
and $\frac{6.50}{x} = \frac{13}{2x} =$ length second piece.

From the conditions, $\frac{5}{x} + 10 : \frac{13}{2x} + 10 :: 5 : 6$
Mult. ext. and means, $\frac{30}{x} + 60 = \frac{65}{2x} + 50$
Clear. of fractions, $60 + 120x = 65 + 100x$
Transposing and uniting, $20x = 5$
Whence, $x = \frac{1}{4}$

Then, $\frac{5}{x} = 20$, length of first.

Also, $\frac{13}{2x} = 26$, length of second.

SECOND SOLUTION.

As the price of each per yard is the same, it follows that the lengths of the two pieces must be in the ratio of their full prices; that is, as 5 to 6.50, or 10 to 13.

Let $10x =$ length of first piece,

and $13x =$ length second piece.

Then, $10x + 10 : 13x + 10 :: 5 : 6$

Mult. ex. and means, $60x + 60 = 65x + 50$

Transp. and uniting, $-5x = -10$

Whence, $x = 2$

Then, $10x = 20$, length of first.

Also, $13x = 26$, length of second.

NOTE. This problem furnishes a good opportunity for the use of negative exponents. If x represent the price per yard, then $5x^{-1}$ and $\frac{13}{2}x^{-1}$ will represent the lengths of each, and

$$5x^{-1} + 10 : \frac{13}{2}x^{-1} + 10 :: 5 : 6,$$

from which $x^{-1} = 4$, $5x^{-1} = 20$, and $\frac{13}{2}x^{-1} = 26$.

52. Let $x =$ A's capital,

and $6300 - x =$ B's capital.

Then, $12x = 8(6300 - x)$

Or, $12x = 50400 - 8x$

Transposing and uniting, $20x = 50400$

Whence, $x = 2520$, A's capital.

Then, $6300 - x = 3780$, B's capital.

54. Let $x =$ no. days he worked,

and $48 - x =$ no. days he was idle.

Then $2x =$ am't rec'd for labor,

and $48 - x =$ am't deducted for board.

From the conditions, $2x - (48 - x) = 42$

Or, $2x - 48 + x = 42$
Transposing and uniting, $3x = 90$
Whence, $x = 30$, no. days he worked.

55. Let $x =$ no. quarts in each.
Then $x - 34 =$ no. quarts left in first,
and $x - 80 =$ no. qts. left in second.

From the conditions, $x - 34 = 2(x - 80)$
Or, $x - 34 = 2x - 160$
Transp. and uniting, $-x = -126$
Or, $x = 126$, no. quarts in each.

56. Let $x =$ no. persons.
Then $\frac{2x}{3} \times 18 = 12x =$ am't rec'd by two thirds,
and $\frac{x}{3} \times 30 = 10x =$ am't rec'd by one third.

Therefore, $12x + 10x = 660$
Or, $22x = 660$
Whence, $x = 30$, no. persons.

57. Let $x =$ weight of the body.
Then $12 + \frac{x}{2} =$ weight of the tail,
and $24 + \frac{3x}{2} =$ weight of the fish.

From the conditions, $x = 12 + 12 + \frac{x}{2} + 26$
Or, $x = 50 + \frac{x}{2}$
Clearing of fractions, $2x = 100 + x$
Whence, $x = 100$, weight of body.
Then, $24 + \frac{3x}{2} = 174$, weight of fish.

58. Let $x =$ rate river flows.
Then $9 + x =$ rate he moves down,
and $9 - x =$ rate he moves up.

Therefore, $9 + x = 2(9 - x)$
Or, $9 + x = 18 - 2x$
Transposing and uniting, $3x = 9$
Whence, $x = 3$, rate river flows.

60. Let $x =$ number of hogs,
and $35 - x =$ number of pigs.

Then, $1250x + 250(35 - x) = 19750$
Dividing by 250, $5x + 35 - x = 79$
Transposing and uniting, $4x = 44$
Whence, $x = 11$, no. hogs.
Then, $35 - x = 24$, no. pigs.

61. Let $x =$ no. men on a side at first.
Then $x^2 + 21 =$ the whole no of men.

Therefore, $(x + 1)^2 = x^2 + 21 + 200$
Or, $x^2 + 2x + 1 = x^2 + 221$
Whence, $2x = 220$
Dividing by 2, $x = 110$
Squaring, $x^2 = 12100$
Then, $x^2 + 21 = 12121$, whole no. men.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

(ART. 172, page 134.)

2. Given $\begin{cases} 2x + 3y = 23 & (1) \\ 5x - 2y = 10 & (2) \end{cases}$

Transp. $3y$ in (1) and div. by 2, $x = \dfrac{23 - 3y}{2}$ (3)

Transp. $2y$ in (2) and div. by 5, $x = \dfrac{10 + 2y}{5}$ (4)

By (3), (4), and Ax. 7, $\dfrac{10 + 2y}{5} = \dfrac{23 - 3y}{2}$ (5)

Clearing of fractions, $20 + 4y = 115 - 15y$ (6)

Transposing and uniting, $19y = 95$ (7)

Dividing by 19, $y = 5$ (8)

Substituting in (4). $x = \dfrac{10 + 10}{5} = 4$ (9)

3. Given $\begin{cases} 4x + y = 34 & (1) \\ x + 4y = 16 & (2) \end{cases}$

From (1), $x = \dfrac{34 - y}{4}$ (3)

From (2), $x = 16 - 4y$ (4)

By Ax. 7, $\dfrac{34 - y}{4} = 16 - 4y$ (5)

Clearing of fractions, $34 - y = 64 - 16y$ (6)

Transposing and uniting, $15y = 30$ (7)

Whence, $y = 2$ (8)

Substituting in (4), $x = 16 - 8 = 8$ (9)

4. Given
$$\begin{cases} 5x - 3y = 9 & (1) \\ 2x + 5y = 16 & (2) \end{cases}$$

From (1), $$x = \frac{9 + 3y}{5} \qquad (3)$$

From (2), $$x = \frac{16 - 5y}{2} \qquad (4)$$

By Ax. 7, $$\frac{9 + 3y}{5} = \frac{16 - 5y}{2} \qquad (5)$$

Clearing of fractions, $18 + 6y = 80 - 25y$ (6)

Transposing and uniting, $31y = 62$ (7)

Whence, $y = 2$ (8)

Substituting in (3); $x = \frac{9 + 6}{5} = 3$ (9)

5. Given
$$\begin{cases} 7x + 3y = 13 & (1) \\ 5x + 2y = 9 & (2) \end{cases}$$

From (1), $$y = \frac{13 - 7x}{3} \qquad (3)$$

From (2), $$y = \frac{9 - 5x}{2} \qquad (4)$$

By Ax. 7, $$\frac{9 - 5x}{2} = \frac{13 - 7x}{3} \qquad (5)$$

Clearing of fractions, $27 - 15x = 26 - 14x$ (6)

Transposing and uniting, $-x = -1$

Or, $x = 1$

Substituting in (4), $y = \frac{9 - 5}{2} = 2$ } Ans.

6. Given
$$\begin{cases} 8x - 7y = -15 & (1) \\ 3y - 6x = -9 & (2) \end{cases}$$

From (1), $$x = \frac{7y - 15}{8} \qquad (3)$$

From (2), $$x = \frac{3y + 9}{6} \qquad (4)$$

By Ax. 7, $$\frac{7y - 15}{8} = \frac{3y + 9}{6} \qquad (5)$$

Multiplying by 24, $21y - 45 = 12y + 36$ (6)
Transposing and uniting, $9y = 81$ (7)
Whence, $y = 9$ (8)
Substituting in (4), $x = \frac{27 + 9}{6} = 6$ (9)

7. Given $\begin{cases} 14x + 6y = 0 & (1) \\ 6x - 46 = 4y & (2) \end{cases}$

From (1), $y = -\frac{14x}{6} = -\frac{7x}{3}$ (3)
From (2), $y = \frac{6x - 46}{4} = \frac{3x - 23}{2}$ (4)
By Ax. 7, $\frac{3x - 23}{2} = -\frac{7x}{3}$ (5)
Clearing of fractions, $9x - 69 = -14x$ (6)
Transposing and uniting, $23x = 69$ (7)
Whence, $x = 3$ (8)
Substituting in (3), $y = -\frac{7 \times 3}{3} = -7$ (9)

8. Given $\begin{cases} \frac{x}{2} + \frac{y}{3} = 7 & (1) \\ \frac{x}{3} + \frac{y}{2} = 8 & (2) \end{cases}$

Clearing (1) of fractions, $3x + 2y = 42$ (3)
Clearing (2) of fractions, $2x + 3y = 48$ (4)
From (3), $x = \frac{42 - 2y}{3}$ (5)
From (4), $x = \frac{48 - 3y}{2}$ (6)
By Ax. 7, $\frac{42 - 2y}{3} = \frac{48 - 3y}{2}$ (7)
Clearing of fractions, $84 - 4y = 144 - 9y$ (8)
Transposing and uniting, $5y = 60$ (9)
Whence, $y = 12$ (10)
Substituting in (5), $x = \frac{42 - 24}{3} = 6$ (11)

(Art. **173**, pp. 135, 136.)

3. Given $\begin{cases} x + 4y = 16 & (1) \\ 4x + y = 34 & (2) \end{cases}$

From (1), $x = 16 - 4y$ (3)

Substituting in (2), $4(16 - 4y) + y = 34$ (4)

Expanding, $64 - 16y + y = 34$ (5)

Transposing and uniting, $-15y = -30$ (6)

Whence, $y = 2$ (7)

Substituting in (3), $x = 16 - 8 = 8$ (8)

4. Given $\begin{cases} x + 2y = 18 & (1) \\ 2x - y = 1 & (2) \end{cases}$

From (1), $x = 18 - 2y$ (3)

Substituting in (2), $2(18 - 2y) - y = 1$ (4)

Expanding, $36 - 4y - y = 1$ (5)

Transposing and uniting, $-5y = -35$ (6)

Whence, $y = 7$ (7)

Substituting in (3), $x = 18 - 14 = 4$ (8)

5. Given $\begin{cases} x + y = 13 & (1) \\ x - y = 3 & (2) \end{cases}$

From (1), $x = 13 - y$ (3)

Substituting in (2), $13 - y - y = 3$ (4)

Transposing and uniting, $-2y = -10$ (5)

Whence, $y = 5$ (6)

Substituting in (3), $x = 13 - 5 = 8$ (7)

6. Given $\begin{cases} \frac{x}{2} - y = 1 & (1) \\ x - \frac{y}{2} = 8 & (2) \end{cases}$

Clearing (1) of fractions, $x - 2y = 2$ (3)

From (3), $x = 2 + 2y$ (4)

Substituting in (2),	$2 + 2y - \frac{y}{2} = 8$	(5)
Clearing (5) of fractions,	$4 + 4y - y = 16$	(6)
Transposing and uniting,	$3y = 12$	(7)
Whence,	$y = 4$	(8)
Substituting in (4),	$x = 2 + 8 = 10$	(9)

7. Given $\begin{cases} 3x + 5y = 40 & (1) \\ x + 2y = 14 & (2) \end{cases}$

From (2),	$x = 14 - 2y$	(3)
Substituting in (1),	$3(14 - 2y) + 5y = 40$	(4)
Expanding,	$42 - 6y + 5y = 40$	(5)
Transposing and uniting,	$-y = -2$	(6)
Whence,	$y = 2$	(7)
Substituting in (3),	$x = 14 - 4 = 10$ } Ans.	(8)

8. Given $\begin{cases} 5x + 3y = 0 & (1) \\ x - y = 8 & (2) \end{cases}$

From (2),	$x = 8 + y$	(3)
Substituting in (1),	$5(8 + y) + 3y = 0$	(4)
Expanding,	$40 + 5y + 3y = 0$	(5)
Transposing and uniting,	$8y = -40$	(6)
Whence,	$y = -5$	(7)
Substituting in (3),	$x = 8 + (-5) = 3$	(8)

9. Given $\begin{cases} 6x + 5y = 77 & (1) \\ 4x - 3y = 7 & (2) \end{cases}$

From (1),	$x = \frac{77 - 5y}{6}$	(3)
Substituting in (2),	$4\left(\frac{77 - 5y}{6}\right) - 3y = 7$	(4)
Expanding,	$\frac{154 - 10y}{3} - 3y = 7$	(5)
Clearing of fractions,	$154 - 10y - 9y = 21$	(6)
Transposing and uniting,	$-19y = -133$	(7)

Whence, $y = 7$ (8)

Substituting in (3), $x = \frac{77 - 35}{6} = 7$ (9)

10. Given $\begin{cases} \frac{x}{3} + \frac{2y}{5} = 6 & (1) \\ \frac{2x}{3} + \frac{y}{5} = 6 & (2) \end{cases}$

From (1), $\frac{x}{3} = 6 - \frac{2y}{5}$ (3)

Substituting in (2), $2\left(6 - \frac{2y}{5}\right) + \frac{y}{5} = 6$ (4)

Expanding, $12 - \frac{4y}{5} + \frac{y}{5} = 6$ (5)

Transposing and uniting, $-\frac{3y}{5} = -6$ (6)

Multiplying by $-\frac{5}{3}$, $y = 10$ (7)

Substituting in (3), $\frac{x}{3} = 6 - \frac{20}{5} = 2$ (8)

Multiplying by 3, $x = 6$ (9)

11. Given $\begin{cases} \frac{x+2}{3} + 8y = 31 & (1) \\ \frac{y+5}{4} + 10x = 192 & (2) \end{cases}$

Clearing (1) of fractions, $x + 2 + 24y = 93$ (3)

Clearing (2) of fractions, $y + 5 + 40x = 768$ (4)

Reducing (3), $x + 24y = 91$ (5)

Reducing (4), $40x + y = 763$ (6)

From (5), $x = 91 - 24y$ (7)

Substituting in (6), $40(91 - 24y) + y = 763$ (8)

Expanding, $3640 - 960y + y = 763$ (9)

Transposing and uniting, $-959y = -2877$ (10)

Whence, $y = 3$ (11)

Substituting in (7), $x = 91 - 72 = 19$ (12)

(Art. 174, pp. 137, 138.)

3. Given $\begin{cases} 4x + 3y = 25 & (1) \\ 12x - 6y = 30 & (2) \end{cases}$

Equation (1),	$4x + 3y = 25$	
Dividing (2) by 2,	$6x - 3y = 15$	(3)
Adding (1) and (3),	$10x = 40$	(4)
Whence,	$x = 4$	(5)
Substituting in (1),	$16 + 3y = 25$	(6)
Transposing and uniting,	$3y = 9$	(7)
Whence,	$y = 3$	(8)

4. Given $\begin{cases} 3x - y = 22 & (1) \\ 2x + 4y = 24 & (2) \end{cases}$

Multiplying (1) by 2,	$6x - 2y = 44$	(3)
Dividing (2) by 2,	$x + 2y = 12$	(4)
Adding (3) and (4),	$7x = 56$	(5)
Whence,	$x = 8$	(6)
Substituting in (1),	$24 - y = 22$	(7)
Transposing and uniting,	$-y = -2$	(8)
Or,	$y = 2$	(9)

5. Given $\begin{cases} x + 8y = 44 & (1) \\ 6x + y = 29 & (2) \end{cases}$

Multiplying (1) by 6,	$6x + 48y = 264$	(3)
Subtracting (2) from (3),	$47y = 235$	(4)
Whence,	$y = 5$	(5)
Substituting in (1),	$x + 40 = 44$	(6)
Transposing and uniting,	$x = 4$	(7)

6. Given $\begin{cases} 23x - 8y = 70 & (1) \\ 8x - 2y = 40 & (2) \end{cases}$

Multiplying (2) by 4,	$32x - 8y = 160$	(3)
Equation (1),	$23x - 8y = 70$	
Subtracting (1) from (3),	$9x = 90$	(4)

Whence,	$x = 10$	(5)
Substituting in (2),	$80 - 2y = 40$	(6)
Transposing and uniting,	$-2y = -40$	(7)
Whence,	$y = 20$	(8)

7. Given $\begin{cases} 4x - 5y = 0 & (1) \\ x - y = 1 & (2) \end{cases}$

Multiplying (2) by 4,	$4x - 4y = 4$	(3)
Equation (1),	$4x - 5y = 0$	
Subtracting (1) from (3),	$y = 4$ Ans.	(4)
Substituting in (2),	$x - 4 = 1$	(5)
Transposing and uniting,	$x = 5$ Ans.	(6)

8. Given $\begin{cases} x + y = 35 & (1) \\ \frac{3x}{7} + \frac{9y}{14} = 18 & (2) \end{cases}$

Dividing (2) by 3,	$\frac{x}{7} + \frac{3y}{14} = 6$	(3)
Clearing (3) of fractions,	$2x + 3y = 84$	(4)
Multiplying (1) by 2,	$2x + 2y = 70$	(5)
Subtracting (5) from (4),	$y = 14$	(6)
Substituting in (1),	$x + 14 = 35$	(7)
Transposing and uniting,	$x = 21$	(8)

9. Given $\begin{cases} \frac{7x}{8} + \frac{4y}{9} = 29 & (1) \\ \frac{11x}{12} - \frac{5y}{6} = 7 & (2) \end{cases}$

Clearing (1) of fractions,	$63x + 32y = 2088$	(3)
Clearing (2) of fractions,	$11x - 10y = 84$	(4)
Multiplying (3) by 5,	$315x + 160y = 10440$	(5)
Multiplying (4) by 16,	$176x - 160y = 1344$	(6)
Adding (5) and (6),	$491x = 11784$	(7)
Whence,	$x = 24$	(8)

Substituting in (4), $264 - 10y = 84$ (9)
Transposing and uniting, $-10y = -180$ (10)
Whence, $y = 18$ (11)

(Art. **175**, pp. 138, 139.)

1\. Given $\begin{cases} 2x + y = 35 & (1) \\ 5x - 3y = 27 & (2) \end{cases}$

From (1), $y = 35 - 2x$ (3)
Substituting in (2), $5x - 3(35 - 2x) = 27$ (4)
Expanding, $5x - 105 + 6x = 27$ (5)
Transposing and uniting, $11x = 132$ (6)
Whence, $x = 12$ (7)
Substituting in (3), $y = 35 - 24 = 11$ (8)

2\. Given $\begin{cases} 5y - 5x = 15 & (1) \\ 3x + 5y = 71 & (2) \end{cases}$

Subtracting (1) from (2), $8x = 56$ (3)
Whence, $x = 7$ (4)
Dividing (1) by 5, $y - x = 3$ (5)
Substituting (4) in (5), $y - 7 = 3$ (6)
Whence, $y = 10$ (7)

3\. Given $\begin{cases} \frac{x}{6} + \frac{y}{4} = 6 & (1) \\ \frac{x}{4} + \frac{y}{6} = 5\frac{2}{3} & (2) \end{cases}$

Clearing (1) of fractions, $2x + 3y = 72$ (3)
Clearing (2) of fractions, $3x + 2y = 68$ (4)
From (3), $y = \frac{72 - 2x}{3}$ (5)
From (4), $y = \frac{68 - 3x}{2}$ (6)
By Ax. 7, $\frac{72 - 2x}{3} = \frac{68 - 3x}{2}$ (7)
Clearing of fractions, $144 - 4x = 204 - 9x$ (8)

Transposing and uniting, $5x = 60$ (9)
Whence, $x = 12$ (10)

Substituting in (5), $y = \frac{72 - 24}{3} = 16$ (11)

4. Given
$$\begin{cases} 3x + 2y = 4 & (1) \\ 4y - 3x + 1 = 0 & (2) \end{cases}$$

Adding (1) and (2), $6y + 1 = 4$ (3)
Transposing and uniting, $6y = 3$ (4)
Whence, $y = \frac{1}{2}$ (5)
Substituting in (1), $3x + 1 = 4$ (6)
Transposing and uniting, $3x = 3$ (7)
Whence, $x = 1$ (8)

5. Given
$$\begin{cases} -4x + 3y = 45 & (1) \\ 2y + 6x = 4 & (2) \end{cases}$$

Multiplying (1) by 3, $-12x + 9y = 135$ (3)
Multiplying (2) by 2, $12x + 4y = 8$ (4)

Adding (3) and (4), $13y = 143$ (5)
Whence, $y = 11$ (6)
Substituting in (2), $22 + 6x = 4$ (7)
Transposing and uniting, $6x = -18$ (8)
Whence, $x = -3$ (9)

6. Given
$$\begin{cases} \frac{x}{2} + y = 35 & (1) \\ \frac{y}{3} + x = 45 & (2) \end{cases}$$

Clearing (1) of fractions, $x + 2y = 70$ (3)
Clearing (2) of fractions, $3x + y = 135$ (4)
From (3), $x = 70 - 2y$ (5)
Substituting in (4), $3(70 - 2y) + y = 135$ (6)
Expanding, $210 - 6y + y = 135$ (7)
Transposing and uniting, $-5y = -75$ (8)

Whence,	$y = 15$	(9)
Substituting in (5),	$x = 70 - 30 = 40$	(10)

7. Given $\begin{cases} 10x = 2 + 2y & (1) \\ 4y = 20 - 4x & (2) \end{cases}$

Reducing (1),	$5x - y = 1$	(3)
Reducing (2),	$x + y = 5$	(4)
Adding (3) and (4),	$6x = 6$	(5)
Whence,	$x = 1$	(6)
Substituting in (4),	$1 + y = 5$	(7)
Whence,	$y = 4$	(8)

8. Given $\begin{cases} x + y = a & (1) \\ x - y = b & (2) \end{cases}$

Adding (1) and (2),	$2x = a + b$	(3)
Whence,	$x = \frac{a + b}{2}$	(4)
Subtracting (2) from (1),	$2y = a - b$	(5)
Whence,	$y = \frac{a - b}{2}$	(6)

9. Given $\begin{cases} 7x - 3y = 26 & (1) \\ 2x - 2y = 6\frac{2}{3} & (2) \end{cases}$

Multiplying (1) by 2,	$14x - 6y = 52$	(3)
Multiplying (2) by 3,	$6x - 6y = 20$	(4)
Subtracting (4) from (3)	$8x = 32$	(5)
Whence,	$x = 4$	(6)
Substituting in (4),	$24 - 6y = 20$	(7)
Transposing and uniting,	$-6y = -4$	(8)
Whence,	$y = \frac{2}{3}$	(9)

10. Given $\begin{cases} \frac{3x}{4} - \frac{y}{2} = 9 & (1) \\ 2x - 2y = 16 & (2) \end{cases}$

Clearing (1) of fractions,	$3x - 2y = 36$	(3)

Subtracting (2) from (3), $x = 20$ (4)
Dividing (2) by 2, $x - y = 8$ (5)

Subtracting (5) from (4), $y = 12$ (6)

11. Given $\begin{cases} \frac{x}{4} + \frac{y}{5} = 5 & (1) \\ \frac{2x}{3} + y = 18 & (2) \end{cases}$

Clearing (1) of fractions, $5x + 4y = 100$ (3)
Clearing (2) of fractions, $2x + 3y = 54$ (4)

From (3), $x = \frac{100 - 4y}{5}$ (5)

From (4), $x = \frac{54 - 3y}{2}$ (6)

By Ax. 7, $\frac{100 - 4y}{5} = \frac{54 - 3y}{2}$ (7)

Clearing of fractions, $200 - 8y = 270 - 15y$ (8)
Transposing and uniting, $7y = 70$ (9)
Whence, $y = 10$ (10)

Substituting in (6), $x = \frac{54 - 30}{2} = 12$ (11)

Note. The last equations are readily solved by multiplying (2) by 4 and subtracting the resulting equation from (3), thus obtaining $\frac{7x}{3} = 28$.

12. Given $\begin{cases} 2y + 79 = 5x & (1) \\ 3x - 7 = 4 + x + y & (2) \end{cases}$

Transp. and uniting in (2), $2x - y = 11$ (3)
From (3), $y = 2x - 11$ (4)
Substituting in (1), $4x - 22 + 79 = 5x$ (5)
Transposing and uniting, $-x = -57$ (6)
Whence, $x = 57$ (7)
Substituting in (4), $y = 114 - 11 = 103$ (8)

13. Given
$$\begin{cases} \dfrac{1}{x}+\dfrac{2}{y}=\dfrac{11}{15} & (1) \\ \dfrac{3}{x}+\dfrac{4}{y}=\dfrac{9}{5} & (2) \end{cases}$$

Multiplying (1) by 3, $\quad \dfrac{3}{x}+\dfrac{6}{y}=\dfrac{11}{5} \qquad (3)$

Subtracting (2) from (3), $\quad \dfrac{2}{y}=\dfrac{2}{5} \qquad (4)$

Dividing by 2, $\quad \dfrac{1}{y}=\dfrac{1}{5} \qquad (5)$

Whence, $\quad y=5 \qquad (6)$

Substituting (4) in (1), $\quad \dfrac{1}{x}+\dfrac{2}{5}=\dfrac{11}{15} \qquad (7)$

Transposing and uniting, $\quad \dfrac{1}{x}=\dfrac{5}{15}=\dfrac{1}{3} \qquad (8)$

Whence, $\quad x=3 \qquad (9)$

14. Given
$$\begin{cases} 2x^{-1}+3y^{-1}=\tfrac{3}{2} & (1) \\ 4x^{-1}+8y^{-1}=\tfrac{10}{3} & (2) \end{cases}$$

Multiplying (1) by 2, $\quad 4x^{-1}+6y^{-1}=3 \qquad (3)$

Subtracting (3) from (2), $\quad 2y^{-1}=\tfrac{1}{3} \qquad (4)$

Dividing by 2, $\quad y^{-1}=\tfrac{1}{6} \qquad (5)$

Multiplying by $6y$, $\quad 6=y \qquad (6)$

Or, $\quad y=6 \qquad (7)$

Substituting (5) in (1), $\quad 2x^{-1}+\tfrac{1}{2}=\tfrac{3}{2} \qquad (8)$

Transposing and uniting, $\quad 2x^{-1}=1 \qquad (9)$

Dividing by 2, $\quad x^{-1}=\tfrac{1}{2} \qquad (10)$

Whence, $\quad x=2 \qquad (11)$

NOTE. We might have divided (2) by 2, instead of multiplying (1) by 2.

It will be observed that examples 13 and 14 are similar to each other, in one case the fractional form being employed, and in the other negative exponents. The pupil, of course, already understands that x^{-1} and y^{-1} are the same as $\dfrac{1}{x}$ and $\dfrac{1}{y}$.

15. Given
$$\begin{cases} \frac{x}{2} + \frac{y}{3} = a & (1) \\ \frac{x}{3} + \frac{y}{4} = b & (2) \end{cases}$$

Clearing (1) of fractions,	$3x + 2y = 6a$	(3)
Clearing (2) of fractions,	$4x + 3y = 12b$	(4)
Multiplying (3) by 4,	$12x + 8y = 24a$	(5)
Multiplying (4) by 3,	$12x + 9y = 36b$	(6)
Subtracting (5) from (6),	$y = 36b - 24a$	(7)
Multiplying (3) by 3,	$9x + 6y = 18a$	(8)
Multiplying (4) by 2,	$8x + 6y = 24b$	(9)
Subtracting (9) from (8),	$x = 18a - 24b$	(10)

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

(ART. 176, pp. 140-145.)

2. Let $x =$ the greater number,
and $y =$ the less number.

Then,	$x + y = 133$	(1)
and	$x - y = 47$	(2)
Adding (1) and (2),	$2x = 180$	(3)
Whence,	$x = 90$	(4)
Subtracting (2) from (1),	$2y = 86$	(5)
Whence,	$y = 43$	(6)

Then, $x + y = 84$ (1)

and $\frac{x}{y} = 13$ (2)

Clearing (2) of fractions, $x = 13y$ (3)

Substituting in (1), $13y + y = 84$ (4)

Uniting terms, $14y = 84$ (5)

Whence, $y = 6$ (6)

Substituting in (3), $x = 13 \times 6 = 78$ (7)

11. Let $x =$ James's age,

and $y =$ John's age.

Then, $x : y :: 3 : 4$ (1)

and $x + 6 : y + 6 :: 5 : 6$ (2)

From (1), $4x = 3y$ (3)

From (2), $6x + 36 = 5y + 30$ (4)

Transposing and uniting in (4), $6x - 5y = -6$ (5)

Multiplying (3) by 3, $12x - 9y = 0$ (6)

Multiplying (5) by 2, $12x - 10y = -12$ (7)

Subtracting (7) from (6), $y = 12$ (8)

Substituting in (3), $4x = 36$ (9)

Whence, $x = 9$ (10)

12. Let $x =$ greater number,

and $y =$ less number.

Then, $x : 24 :: x + y : 42$ (1)

and $x - y : 6 :: 4 : 3$ (2)

From (1), $42x = 24x + 24y$ (3)

From (2), $3x - 3y = 24$ (4)

Reducing (3), $3x - 4y = 0$ (5)

Subtracting (5) from (4), $y = 24$ (6)

Substituting in (5), $3x = 96$ (7)

Whence, $x = 32$ (8)

14. Let $x =$ greater part,
and $y =$ less part.

Then,	$x + y = 72$	(1)
and	$3x - 2y = 121$	(2)
Multiplying (1) by 2,	$2x + 2y = 144$	(3)
Adding (2) and (3),	$5x = 265$	(4)
Whence,	$x = 53$	(5)
Substituting in (1),	$53 + y = 72$	(6)
Whence,	$y = 19$	(7)

15. Let $x =$ no. men at \$0.90,
and $y =$ no. men at \$1.50.

Then,	$x + y = 50$	(1)
and	$90x + 150y = 4800$	(2)
Multiplying (1) by 3,	$3x + 3y = 150$	(3)
Dividing (2) by 30,	$3x + 5y = 160$	(4)
Subtracting (3) from (4),	$2y = 10$	(5)
Whence,	$y = 5$	(6)
Substituting in (1),	$x + 5 = 50$	(7)
Whence,	$x = 45$	(8)

16. Let $x =$ wages of each man,
and $y =$ wages of each woman.

Then,	$5x + 7y = 1640$	(1)
and	$7x - 6y = 400$	(2)
Multiplying (1) by 7,	$35x + 49y = 11480$	(3)
Multiplying (2) by 5,	$35x - 30y = 2000$	(4)
Subtracting (4) from (3),	$79y = 9480$	(5)
Whence,	$y = 120$	(6)
Substituting in (1),	$5x + 840 = 1640$	(7)
Transposing and uniting,	$5x = 800$	(8)

Whence, $x = 160$ (9)

NOTE. In the last two solutions, the sums of money are expressed in cents throughout.

17. Let $x =$ the numerator,
and $y =$ the denominator.
Therefore $\frac{x}{y} =$ the fraction.

Then, $$\frac{x+4}{y} = \frac{1}{2} \qquad (1)$$

and $$\frac{x}{y+7} = \frac{1}{5} \qquad (2)$$

From (1), $$2x - y = -8 \qquad (3)$$

From (2), $$5x - y = 7 \qquad (4)$$

Subtracting (3) from (4), $$3x = 15 \qquad (5)$$

Whence, $$x = 5 \qquad (6)$$

Substituting in (4), $$25 - y = 7 \qquad (7)$$

Whence, $$-y = -18 \qquad (8)$$

Or, $$y = 18 \qquad (9)$$

Therefore, $$\frac{x}{y} = \frac{5}{18} \qquad (10)$$

19. Let $x =$ income tax,
and $y =$ assessed tax.

Then, $$x + y = 30 \qquad (1)$$

and $$x + \frac{20x}{100} + y - \frac{25y}{100} = 32\tfrac{1}{8} \qquad (2)$$

From (2), $$\frac{24x}{20} + \frac{15y}{20} = \frac{257}{8} \qquad (3)$$

Clearing (3) of fractions, $$48x + 30y = 1285 \qquad (4)$$

Multiplying (1) by 30, $$30x + 30y = 900 \qquad (5)$$

Subtracting (5) from (4), $$18x = 385 \qquad (6)$$

Whence, $$x = 21\tfrac{7}{18} \qquad (7)$$

Substituting in (1), $$21\tfrac{7}{18} + y = 30 \qquad (8)$$

Whence, $$y = 8\tfrac{11}{18} \qquad (9)$$

20. Let $x =$ the first quantity,
and $y =$ the second quantity.

Then,	$x + a = m y$	(1)
and	$y + b = n x$	(2)
From (1),	$-x + m y = a$	(3)
Multiplying (2) by m,	$m n x - m y = b m$	(4)
Adding (3) and (4),	$m n x - x = a + b m$	(5)
Whence,	$x = \frac{a + b m}{m n - 1}$	(6)
Multiplying (3) by n,	$-n x + m n y = a n$	(7)
From (2),	$n x - y = b$	(8)
Adding (7) and (8),	$m n y - y = b + a n$	(9)
Whence,	$y = \frac{b + a n}{m n - 1}$	(10)

21. Let $x =$ A's money,
and $y =$ B's money.

Then,	$x = \frac{4}{9} y$	(1)
and	$x + 10 = y - 10$	(2)
Substituting (1) in (2),	$\frac{4}{9} y + 10 = y - 10$	(3)
Clearing (3) of fractions,	$4 y + 90 = 9 y - 90$	(4)
Transposing and uniting,	$-5 y = -180$	(5)
Whence,	$y = 36$	(6)
Substituting in (1),	$x = \frac{4}{9} \times 36 = 16$	(7)

SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

(Art. 177, pp. 147, 148.)

3. Given

$$\begin{cases} x + 2y + z = 24 & (1) \\ 2x + y + 3z = 38 & (2) \\ 3x + 3y + 2z = 46 & (3) \end{cases}$$

Multiplying (1) by 3, $3x + 6y + 3z = 72$ (4)

Equation (2), $2x + y + 3z = 38$

Subtracting (2) from (4), $x + 5y = 34$ (5)

Multiplying (1) by 2, $2x + 4y + 2z = 48$ (6)

Equation (3), $3x + 3y + 2z = 46$

Subtracting (3) from (6), $-x + y = 2$ (7)

Equation (5), $x + 5y = 34$

Adding (7) and (5), $6y = 36$ (8)

Whence, $y = 6$ (9)

Substituting in (5), $x + 30 = 34$ (10)

Whence, $x = 4$ (11)

Substituting (9) and (11) in (1), $4 + 12 + z = 24$ (12)

Whence, $z = 8$ (13)

4. Given $\begin{cases} 4x + 2y - z = 26 & (1) \\ 5x + 2y - 3z = 16 & (2) \\ 2x - y + 2z = 23 & (3) \end{cases}$

FIRST SOLUTION.

From (1), $z = 4x + 2y - 26$ (4)

Subst. in (2), $5x + 2y - 3(4x + 2y - 26) = 16$ (5)

Subst. in (3), $2x - y + 2(4x + 2y - 26) = 23$ (6)

Expanding (5), $5x + 2y - 12x - 6y + 78 = 16$ (7)

Expanding (6), $2x - y + 8x + 4y - 52 = 23$ (8)

From (7), $7x + 4y = 62$ (9)

From (8), $10x + 3y = 75$ (10)

From (9), $y = \frac{62 - 7x}{4}$ (11)

Substituting in (10), $10x + 3\left(\frac{62 - 7x}{4}\right) = 75$ (12)

Expand. and clear. fract., $40x + 186 - 21x = 300$ (13)

Transposing and uniting, $19x = 114$ (14)

Whence, $x = 6$ (15)

Substituting in (11), $y = \frac{62 - 42}{4} = 5$ (16)

Substituting in (4), $z = 24 + 10 - 26 = 8$ (17)

SECOND SOLUTION.

Subtracting (2) from (1),	$-x+2z=10$	(4)
Multiplying (3) by 2,	$4x-2y+4z=46$	(5)
Adding (2) and (5),	$9x+z=62$	(6)
Multiplying (6) by 2,	$18x+2z=124$	(7)
Subtracting (4) from (7),	$19x=114$	(8)
Whence,	$x=6$	(9)
Substituting in (6),	$54+z=62$	(10)
Whence,	$z=8$	(11)
Substituting in (3),	$12-y+16=23$	(12)
Whence,	$y=5$	(13)

NOTE. The first solution is given to illustrate elimination by substitution when there are three unknown quantities.

5. Given $\begin{cases} x+y+z=33 & (1) \\ y-x+z=23 & (2) \\ z-x-y=1 & (3) \end{cases}$

Subtracting (2) from (1),	$2x=10$	(4)
Whence,	$x=5$	(5)
Subtracting (3) from (2),	$2y=22$	(6)
Whence,	$y=11$	(7)
Adding (1) and (3),	$2z=34$	(8)
Whence,	$z=17$	(9)

NOTE. The above equations are peculiar, from the fact that two letters may be eliminated at once. This only happens when the coefficients of two letters may be made alike in two equations, and the signs either alike in the case of both letters, or different in the case of both.

7. Given $\begin{cases} x+y+z=13 & (1) \\ u+x+y=17 & (2) \\ u+x+z=18 & (3) \\ u+y+z=21 & (4) \end{cases}$

Assume $s=u+x+y+z$ (5)

From (1), $s - u = 13$ (6)
From (2), $s - z = 17$ (7)
From (3), $s - y = 18$ (8)
From (4), $s - x = 21$ (9)

Adding, and substituting (5), $4s - s = 69$ (10)
Or, $3s = 69$ (11)
Whence, $s = 23$ (12)
Substituting in (6), $23 - u = 13$ (13)
Whence, $u = 10$ (14)
Substituting in (7), $23 - z = 17$ (15)
Whence, $z = 6$ (16)
Substituting in (8), $23 - y = 18$ (17)
Whence, $y = 5$ (18)
Substituting in (9), $23 - x = 21$ (19)
Whence, $x = 2$ (20)

The above equations may also be solved as follows:

Adding (1), (2), (3), and (4), $3u + 3x + 3y + 3z = 69$ (5)
Dividing by 3, $u + x + y + z = 23$ (6)
Equation (1), $x + y + z = 13$

Subtracting (1) from (6), $u = 10$ (7)
Subtracting (2) from (6), $z = 6$ (8)
Subtracting (3) from (6), $y = 5$ (9)
Subtracting (4) from (6), $x = 2$ (10)

8. Given $\left\{\begin{array}{l} 7x - 3y - z = 12 \quad (1) \\ x + 2y + 3z = 17 \quad (2) \\ 4x - y + 2z = 13 \quad (3) \end{array}\right.$

From (1), $z = 7x - 3y - 12$ (4)

From (2), $z = \frac{17 - x - 2y}{3}$ (5)

From (3), $z = \frac{13 - 4x + y}{2}$ (6)

Comparing (4) and (5), $7x - 3y - 12 = \frac{17 - x - 2y}{3}$ (7)

Comparing (4) and (6), $7x - 3y - 12 = \frac{13 - 4x + y}{2}$. (8)

Clear. (7) of fract., $21x - 9y - 36 = 17 - x - 2y$ (9)

Clear. (8) of fract., $14x - 6y - 24 = 13 - 4x + y$ (10)

Transp. and uniting in (9), $22x - 7y = 53$ (11)

Transp. and unit. in (10), $18x - 7y = 37$ (12)

Subtracting (12) from (11), $4x = 16$ (13)

Whence, $x = 4$ (14)

Substituting in (12), $72 - 7y = 37$ (15)

Transposing and uniting, $-7y = -35$ (16)

Whence, $y = 5$ (17)

Substitut. in (4), $z = 28 - 15 - 12 = 1$ (18)

Note. In eliminating the last time, the method of comparison is abandoned, because there is so good an opportunity to eliminate y by subtraction.

9. Given $\begin{cases} x + y - z = 0 & (1) \\ x + z - y = 2 & (2) \\ y + z - x = 4 & (3) \end{cases}$

Adding (1), (2), and (3), $x + y + z = 6$ (4)

Subtracting (1) from (4), $2z = 6$ (5)

Whence, $z = 3$ (6)

Subtracting (2) from (4), $2y = 4$ (7)

Whence, $y = 2$ (8)

Subtracting (3) from (4), $2x = 2$ (9)

Whence, $x = 1$ (10)

Note. A still simpler plan for elimination would be to add the original equations, taken two and two. Thus (1) + (2) gives the value of x, (1) + (3) gives y, and (2) + (3) gives z.

10. Given $\begin{cases} \frac{x}{a} + \frac{y}{b} = 1 & (1) \\ \frac{x}{a} + \frac{z}{c} = 1 & (2) \\ \frac{y}{b} + \frac{z}{c} = 1 & (3) \end{cases}$

Subtracting (3) from (2), $\frac{x}{a}-\frac{y}{b}=0$ (4)

Equation (1), $\frac{x}{a}+\frac{y}{b}=1$

Subtracting (4) from (1), $\frac{2y}{b}=1$ (5)

Whence, $y=\frac{b}{2}$ (6)

Adding (4) and (1), $\frac{2x}{a}=1$ (7)

Whence, $x=\frac{a}{2}$ (8)

Substituting in (2), $\frac{1}{2}+\frac{z}{c}=1$ (9)

Whence, $z=\frac{c}{2}$ (10)

11. Given $\begin{cases} x+y=a & (1) \\ x+z=b & (2) \\ y+z=c & (3) \end{cases}$

Subtracting (3) from (2), $x-y=b-c$ (4)
Adding (1) and (4), $2x=a+b-c$ (5)
Whence, $x=\frac{1}{2}(a+b-c)$ (6)
Subtracting (4) from (1), $2y=a-b+c$ (7)
Whence, $y=\frac{1}{2}(a+c-b)$ (8)
Substituting in (3), $\frac{a+c-b}{2}+z=c$ (9)
Clearing of fractions, etc. $2z=c-a+b$ (10)
Whence, $z=\frac{1}{2}(b+c-a)$ (11)

Note. Examples 10 and 11 may also be solved by subtracting each equation from half the sum of the three.

12. Given $\begin{cases} \frac{x}{4}+\frac{y}{3}+\frac{z}{2}=9 & (1) \\ \frac{2x}{3}-\frac{y}{2}+\frac{3z}{4}=11 & (2) \\ \frac{3x}{4}+\frac{2y}{3}-\frac{z}{2}=9 & (3) \end{cases}$

Adding (1) and (3),	$x + y = 18$	(4)
Clearing (2) of fractions,	$8x - 6y + 9z = 132$	(5)
Clearing (3) of fractions,	$9x + 8y - 6z = 108$	(6)
Multiplying (5) by 2,	$16x - 12y + 18z = 264$	(7)
Multiplying (6) by 3,	$27x + 24y - 18z = 324$	(8)
Adding (7) and (8),	$43x + 12y = 588$	(9)
Multiplying (4) by 12,	$12x + 12y = 216$	(10)
Subtracting (10) from (9),	$31x = 372$	(11)
Whence,	$x = 12$	(12)
Substituting in (4),	$12 + y = 18$	(13)
Whence,	$y = 6$	(14)
Substituting in (1),	$3 + 2 + \frac{z}{2} = 9$	(15)
Transposing and uniting,	$\frac{z}{2} = 4$	(16)
Whence,	$z = 8$	(17)

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

(Art. 178, pp. 149 – 151.)

1. Let x = price of an orange,
y = price of an apple,
and z = price of a pear.

Then,	$3x + y + 2z = 14$	(1)
	$4x + 3y + z = 17$	(2)
and	$x + 4y + 3z = 13$	(3)
Multiplying (2) by 2,	$8x + 6y + 2z = 34$	(4)
Subtracting (1) from (4),	$5x + 5y = 20$	(5)
Dividing (5) by 5,	$x + y = 4$	(6)
Multiplying (2) by 3,	$12x + 9y + 3z = 51$	(7)
Equation (3),	$x + 4y + 3z = 13$	

Subtracting (3) from (7), $11x + 5y = 38$ (8)

Equation (5), $5x + 5y = 20$

Subtracting (5) from (8), $6x = 18$ (9)

Whence, $x = 3$ (10)

Substituting in (6), $3 + y = 4$ (11)

Whence, $y = 1$ (12)

Substituting in (2), $12 + 3 + z = 17$ (13)

Whence, $z = 2$ (14)

2. Let $x =$ Mary's part,

$y =$ Isabel's part,

$z =$ Jane's part,

and $u =$ Ellen's part.

Then, $x + y + z + u = 100$ (1)

$2y + 3u = 160$ (2)

$3x + 2z = 90$ (3)

and $2x + u = 60$ (4)

Adding (2) and (3), $3x + 2y + 2z + 3u = 250$ (5)

Mult. (1) by 2, $2x + 2y + 2z + 2u = 200$ (6)

Subtracting (6) from (5), $x + u = 50$ (7)

Equation (4), $2x + u = 60$

Subtracting (7) from (4), $x = 10$ (8)

Substituting in (7), $10 + u = 50$ (9)

Whence, $u = 40$ (10)

Substituting in (2), $2y + 120 = 160$ (11)

Transposing and uniting, $2y = 40$ (12)

Whence, $y = 20$ (13)

Substituting in (1), $10 + 20 + z + 40 = 100$ (14)

Whence, $z = 30$ (15)

3. Given
$$\begin{cases} \frac{7x}{16} + \frac{12y}{16} + \frac{4z}{16} = 8 & (1) \\ \frac{3x}{16} + \frac{3y}{16} + \frac{7z}{16} = \frac{15}{4} & (2) \\ \frac{6x}{16} + \frac{y}{16} + \frac{5z}{16} = \frac{17}{4} & (3) \end{cases}$$

Clearing (1) of fractions,	$7x + 12y + 4z = 128$	(4)
Clearing (2) of fractions,	$3x + 3y + 7z = 60$	(5)
Clearing (3) of fractions,	$6x + y + 5z = 68$	(6)
Multiplying (5) by 4,	$12x + 12y + 28z = 240$	(7)
Subtracting (4) from (7),	$5x + 24z = 112$	(8)
Multiplying (6) by 3,	$18x + 3y + 15z = 204$	(9)
Subtracting (5) from (9),	$15x + 8z = 144$	(10)
Multiplying (10) by 3,	$45x + 24z = 432$	(11)
Subtracting (8) from (11),	$40x = 320$	(12)
Whence,	$x = 8$	(13)
Substituting in (10),	$120 + 8z = 144$	(14)
Transposing and uniting,	$8z = 24$	(15)
Whence,	$z = 3$	(16)
Substituting in (6),	$48 + y + 15 = 68$	(17)
Whence,	$y = 5$	(18)

4. Let x = price of the chaise,
y = price of the horse,
and z = price of the harness.

Then,	$x + y + z = 400$	(1)
	$x = 4z$	(2)
and	$z = \frac{y}{3}$	(3)
Clearing (3) of fractions,	$y = 3z$	(4)
Subs. (2) and (4) in (1),	$4z + 3z + z = 400$	(5)
Or,	$8z = 400$	(6)
Whence,	$z = 50$	(7)
Substituting in (2),	$x = 4 \times 50 = 200$	(8)
Substituting in (4),	$y = 3 \times 50 = 150$	(9)

NOTE. It will be seen that this problem can be readily solved by the use of only one unknown quantity.

5. Let $x =$ the first number,
$y =$ the second number,
and $z =$ the third number.

Then, $x + y + z = 324$ (1)
$y - x = z - y$ (2)
and $x : z :: 5 : 7$ (3)

Transposing in (2), $2y - x - z = 0$ (4)
Adding (4) and (1), $3y = 324$ (5)
Whence, $y = 108$ (6)
Substituting in (1), $x + 108 + z = 324$ (7)
Whence, $x + z = 216$ (8)
Multiplying (8) by 5, $5x + 5z = 1080$ (9)
From (3), $7x - 5z = 0$ (10)

Adding (9) and (10), $12x = 1080$ (11)
Whence, $x = 90$ (12)
Substituting in (8), $90 + z = 216$ (13)
Whence, $z = 126$ (14)

6. Let $x =$ the husband's age,
$y =$ the wife's age,
and $z =$ the son's age.

Then, $x + z = y + 12$ (1)
$y + z = x + 8$ (2)
and $x + y + z = 92$ (3)

Transposing in (1), $x - y + z = 12$ (4)
Subtracting (4) from (3), $2y = 80$ (5)
Whence, $y = 40$ (6)
Transposing in (2), $-x + y + z = 8$ (7)
Subtracting (7) from (3), $2x = 84$ (8)
Whence, $x = 42$ (9)
Adding (4) and (7), $2z = 20$ (10)
Whence, $z = 10$ (11)

7. Let $x =$ no. bush. wheat,

$y =$ no. bush. barley,

and $z =$ no. bush. oats.

Then, $x + y + z = 146$ (1)

$y - x = 15$ (2)

and $x + y = z$ (3)

Substituting (3) in (1), $2z = 146$ (4)

Whence, $z = 73$ (5)

Therefore, from (3), $x + y = 73$ (6)

Equation (2), $-x + y = 15$

Adding (6) and (2), $2y = 88$ (7)

Whence, $y = 44$ (8)

Subtracting (2) from (6), $2x = 58$ (9)

Whence, $x = 29$ (10)

8. Let x, y, and z represent the number of days in which A, B, and C, respectively, could perform the whole work alone. Then $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, or x^{-1}, y^{-1}, and z^{-1}, will represent the part which each can perform in one day. Also, A and B together can do $\frac{1}{8}$ of the work in one day, A and C $\frac{1}{9}$, and B and C $\frac{1}{10}$.

Hence, $x^{-1} + y^{-1} = \frac{1}{8}$ (1)

$x^{-1} + z^{-1} = \frac{1}{9}$ (2)

and $y^{-1} + z^{-1} = \frac{1}{10}$ (3)

Subtracting (2) from (1), $y^{-1} - z^{-1} = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$ (4)

Adding (3) and (4), $2y^{-1} = \frac{1}{10} + \frac{1}{72} = \frac{41}{360}$ (5)

Dividing by 2, $y^{-1} = \frac{41}{720}$ (6)

Whence, $y = \frac{720}{41} = 17\frac{23}{41}$ (7)

Subtracting (4) from (3), $2z^{-1} = \frac{1}{10} - \frac{1}{72} = \frac{62}{720}$ (8)

Dividing by 2, $z^{-1} = \frac{31}{720}$ (9)

Whence, $z = \frac{720}{31} = 23\frac{7}{31}$ (10)

Substituting (6) in (1), $x^{-1} + \frac{41}{720} = \frac{1}{8}$ (11)

Transposing, $x^{-1} = \frac{1}{8} - \frac{41}{720} = \frac{49}{720}$ (12)

Whence, $x = \frac{720}{49} = 14\frac{34}{49}$ (13)

Note. Elimination may also be conveniently effected, and the value of each unknown quantity found, by subtracting each of the original equations from half the sum of the three equations.

9. Let $x =$ digit in hundreds' place,
$y =$ digit in tens' place,
and $z =$ digit in units' place.
Then $100x + 10y + z =$ the number.

From the conditions, $x + y + z = 9$ (1)

$100x + 10y + z - 198 = 100z + 10y + x$ (2)

and $\dfrac{100x + 10y + z}{x} = 108$ (3)

Transposing and uniting in (2), $99x - 99z = 198$ (4)

Dividing (4) by 99, $x - z = 2$ (5)

Clearing (3) of fractions, $100x + 10y + z = 108x$ (6)

Or, $-8x + 10y + z = 0$ (7)

Multiplying (1) by 10, $10x + 10y + 10z = 90$ (8)

Subtracting (7) from (8), $18x + 9z = 90$ (9)

Dividing (9) by 9, $2x + z = 10$ (10)

Equation (5), $x - z = 2$

Adding (10) and (5), $3x = 12$ (11)

Whence, $x = 4$ (12)

Substituting in (10), $8 + z = 10$ (13)

Whence, $z = 2$ (14)

Substituting in (1), $4 + y + 2 = 9$ (15)

Whence, $y = 3$ (16)

Therefore, $100x + 10y + z = 432$ (17)

10. Let x, y, and z denote the number of days in which A, B, and C, respectively, can reap the field alone. Then, $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, or x^{-1}, y^{-1}, and z^{-1} will denote the part of the field that each can reap in one day. Also, A and B together can reap $\frac{1}{a}$ of the field in one day, A and C $\frac{1}{b}$, and B and C $\frac{1}{c}$.

Hence, $$x^{-1}+y^{-1}=a^{-1} \quad (1)$$
$$x^{-1}+z^{-1}=b^{-1} \quad (2)$$
and $$y^{-1}+z^{-1}=c^{-1} \quad (3)$$

Adding (1), (2), and (3),
$$2\,x^{-1}+2\,y^{-1}+2\,z^{-1}=a^{-1}+b^{-1}+c^{-1} \quad (4)$$
Multiplying (3) by 2, $$2\,y^{-1}+2\,z^{-1}=2\,c^{-1} \quad (5)$$
Subtracting (5) from (4), $$2\,x^{-1}=a^{-1}+b^{-1}-c^{-1} \quad (6)$$
Multiplying by $a\,b\,c\,x$ (Art. 153, Note 3),
$$2\,a\,b\,c=(bc+ac-ab)\,x \quad (7)$$
Whence, $$x=\frac{2\,a\,b\,c}{ac+bc-ab} \quad (8)$$
Equation (4), $$2\,x^{-1}+2\,y^{-1}+2\,z^{-1}=a^{-1}+b^{-1}+c^{-1}$$
Multiplying (2) by 2, $$2\,x^{-1}+2\,z^{-1}=2\,b^{-1} \quad (9)$$
Subtracting (9) from (4), $$2\,y^{-1}=a^{-1}-b^{-1}+c^{-1} \quad (10)$$
Multiplying by $a\,b\,c\,y$, $$2\,a\,b\,c=(b\,c-a\,c+a\,b)\,y \quad (11)$$
Whence, $$y=\frac{2\,a\,b\,c}{a\,b+b\,c-a\,c} \quad (12)$$
Equation (4), $$2\,x^{-1}+2\,y^{-1}+2\,z^{-1}=a^{-1}+b^{-1}+c^{-1}$$
Multiplying (1) by 2, $$2\,x^{-1}+2\,y^{-1}=2\,a^{-1} \quad (13)$$
Subtracting (13) from (4), $$2\,z^{-1}=b^{-1}+c^{-1}-a^{-1} \quad (14)$$
Multiplying by $a\,b\,c\,z$, $$2\,a\,b\,c=(a\,c+a\,b-b\,c)\,z \quad (15)$$
Whence, $$z=\frac{2\,a\,b\,c}{a\,b+a\,c-b\,c} \quad (16)$$

NOTE. Equations (6), (10), and (14) may be readily obtained by the method adopted in the solution of Ex. 11, Art. 177, or Prob. 8 of this Art.

INVOLUTION.

(ART. 185, pp. 155, 156.)

7. Ans. $243\, a^{10} x^{10}$. | 15. Ans. $16\, a^4 x^8$.

(ART. 186, pp. 156, 157.)

8. Ans. $\dfrac{a^{10} c^5 d^{15}}{32\, b^{10}}$.

(ART. 187, pp. 157, 158.)

In these examples, as well as the preceding ones, there are three points to be considered: the *sign* of the power, which is determined by Arts. 182, 183, the *coefficient*, and the *exponents*, both of which are determined by the rule found in Art. 190. The signs of the exponents must be determined by the rules for ordinary multiplication.

It has already been shown that the negative exponent expresses an unperformed division, and that a quantity containing a negative exponent is, in reality, the same as a fraction.

3. The answer to this example cannot be both + and — for the same value of n; but if n is *even*, it is +, while if n is *odd*, it is —.

5. Multiplying the exponents by -2 changes their signs.

6. The answer is negative because the original quantity is negative and the exponent -3 is an odd number.

7. The result is positive because the exponent -4 is an even number. The coefficient becomes

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}.$$

8. If the negative exponent is removed, the answer becomes $\frac{c^{6n}}{a^{6n} b^{6m}}$.

The negative exponents are also readily removed in examples 9 and 10, by transferring factors according to Art. 126.

POWERS OF BINOMIALS.

(Art. **200**, pp. 163, 164.)

In writing any power of a binomial, the student has only to apply the principles found in Arts. 191 – 200.

4. Ans. $a^4 + 4a^3y + 6a^2y^2 + 4ay^3 + y^4$.

6. The coefficients of the seventh power are thus obtained: 1, 7, $\frac{7 \times 6}{2} = 21$, $\frac{21 \times 5}{3} = 35$, $\frac{35 \times 4}{4} = 35$, $\frac{35 \times 3}{5} = 21$, &c.

7. The first form of the answer, according to the Binomial Theorem, would be $a^3 + 3a^2 1 + 3a 1^2 + 1^3$.

8. $(1 - x)^6 = (1)^6 - 6(1)^5 x + 15(1)^4 x^2 - 20(1)^3 x^3 + 15(1)^2 x^4 - 6(1) x^5 + x^6$. The powers of 1, being 1, have no effect as factors, and can therefore be omitted.

9. The coefficients of the eighth power are thus obtained: 1, 8, $\frac{8 \times 7}{2} = 28$, $\frac{28 \times 6}{3} = 56$, $\frac{56 \times 5}{4} = 70$, $\frac{70 \times 4}{5} = 56$, &c.

(Art. **201**, pp. 164, 165.)

2. $(3a + 2b)^4 = (3a)^4 + 4(3a)^3(2b) + 6(3a)^2(2b)^2 + 4(3a)(2b)^3 + (2b)^4 = 81a^4 + 216a^3b + 216a^2b^2 + 96ab^3 + 16b^4$.

3. $(2a-3x)^3=(2a)^3-3(2a)^2(3x)+3(2a)(3x)^2-(3x)^3$
$=8a^3-36a^2x+54ax^2-27x^3.$

4. $(1+3x)^4=(1)^4+4(1)^3(3x)+6(1)^2(3x)^2$
$+4(1)(3x)^3+(3x)^4=1+12x+54x^2+108x^3+81x^4.$

5. $(a^2+b^2)^3=(a^2)^3+3(a^2)^2(b^2)+3(a^2)(b^2)^2+(b^2)^3$
$=a^6+3a^4b^2+3a^2b^4+b^6.$

6. $(3x-5)^3=(3x)^3-3(3x)^2 5+3(3x)5^2-5^3$
$=27x^3-135x^2+225x-125$, Ans.

7. $(3xy-a)^2=(3xy)^2-2(3xy)a+a^2$
$=9x^2y^2-6axy+a^2.$

8. $(\frac{1}{2}ab+c)^2=(\frac{1}{2}ab)^2+2(\frac{1}{2}ab)c+c^2$
$=\frac{1}{4}a^2b^2+abc+c^2.$

9. $\left(x-\frac{p}{2}\right)^2=x^2-2x\left(\frac{p}{2}\right)+\left(\frac{p}{2}\right)^2=x^2-px+\frac{p^2}{4}.$

10. $\left(3x+\frac{1}{x}\right)^2=(3x)^2+2(3x)\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)^2=9x^2+6+\frac{1}{x^2}.$

11. $(a+a^{-1})^3=a^3+3a^2(a^{-1})+3a(a^{-1})^2+(a^{-1})^3$
$=a^3+3a+3a^{-1}+a^{-3}.$

NOTE. It will be observed that negative exponents might be employed in the 10th, or the fractional form in the 11th.

12. $(x^2+3y^2)^5=(x^2)^5+5(x^2)^4(3y^2)+10(x^2)^3(3y^2)^2$
$+10(x^2)^2(3y^2)^3+5(x^2)(3y^2)^4+(3y^2)^5=x^{10}+15x^8y^2$
$+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}.$

13. $(\frac{1}{2}x-\frac{2}{3}y)^3=(\frac{1}{2}x)^3-3(\frac{1}{2}x)^2(\frac{2}{3}y)+3(\frac{1}{2}x)(\frac{2}{3}y)^2$
$-(\frac{2}{3}y)^3=\frac{1}{8}x^3-\frac{1}{2}x^2y+\frac{2}{3}xy^2-\frac{8}{27}y^3.$

14. $(x^2-2x)^6=(x^2)^6-6(x^2)^5(2x)+15(x^2)^4(2x)^2$
$-20(x^2)^3(2x)^3+15(x^2)^2(2x)^4-6(x^2)(2x)^5+(2x)^6$
$=x^{12}-12x^{11}+60x^{10}-160x^9+240x^8-192x^7+64x^6.$

EVOLUTION.

SQUARE ROOT OF NUMBERS.

(Art. 215, page 171.)

(3.)

```
       611524 | 782
       49
148 | 1215
    | 1184
 1562 | 3124
      | 3124
```

(4.)

```
     56644 | 238
     4
43 | 166
   | 129
468 | 3744
    | 3744
```

(5.)

```
     6561 | 81, Ans.
     64
161 | 161
    | 161
```

(6.)

```
     2116 | 46
     16
86 | 516
   | 516
```

(7.)

```
     10246401 | 3201
      9
62 | 124
   | 124
6401 | 6401
     | 6401
```

(8.)

```
     16.2409 | 4.03
     16
803 | 2409
    | 2409
```

(9.)

```
      .9409 | .97
       81
187 | 1309
    | 1309
```

(10.)

```
  .0081 | .09
     81
```

(11.)

$$\begin{array}{rl|l}
 & .0\dot{0}6\dot{0}0\dot{0}0\dot{0} & .0774+\\
 & 49 & \\ \hline
147 & 1100 & \\
 & 1029 & \\ \hline
1544 & 7100 & \\
 & 6176 &
\end{array}$$

(12.)

$$\begin{array}{rl|l}
 & 1\dot{2}.0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0} & 3.464101+\\
 & 9 & \\ \hline
64 & 300 & \\
 & 256 & \\ \hline
686 & 4400 & \\
 & 4116 & \\ \hline
6924 & 28400 & \\
 & 27696 & \\ \hline
69281 & 70400 & \\
 & 69281 & \\ \hline
6928201 & 11190000 & \\
 & 6928201 &
\end{array}$$

(13.)

$$\begin{array}{rl|l}
 & .0\dot{0}0\dot{0}0\dot{1}2\dot{3}2\dot{1} & .00111\\
 & 1 & \\ \hline
21 & 23 & \\
 & 21 & \\ \hline
221 & 221 & \\
 & 221 &
\end{array}$$

(Art. 216, page 172.)

2. $\sqrt{\frac{16}{169}} = \frac{\sqrt{16}}{\sqrt{169}} = \frac{4}{13}$.

3. $\sqrt{\frac{484}{729}} = \frac{\sqrt{484}}{\sqrt{729}} = \frac{22}{27}$.

4. $\sqrt{2\frac{47}{121}} = \sqrt{\frac{289}{121}} = \frac{\sqrt{289}}{\sqrt{121}} = \frac{17}{11} = 1\frac{6}{11}$.

5. $\sqrt{\frac{8}{18}}=\sqrt{\frac{4}{9}}=\frac{\sqrt{4}}{\sqrt{9}}=\frac{2}{3}.$

6. $\sqrt{\frac{1899}{10339}}=\sqrt{\frac{1899\div 211}{10339\div 211}}=\sqrt{\frac{9}{49}}=\frac{\sqrt{9}}{\sqrt{49}}=\frac{3}{7}.$

(7.)

$\frac{9}{13}=.6\,9\,2\,3\,0\,8$, nearly.

$$
\begin{array}{r|l}
 & .6\dot{9}2\dot{3}0\dot{8} \;\big|\; .832+ \\
 & 64 \\
\hline
163 & 523 \\
 & 489 \\
\hline
1662 & 3408 \\
 & 3324 \\
\hline
\end{array}
$$

(8.)

$\frac{16}{1031}=.0\,1\,5\,5\,1\,8\,9\,1+$

$$
\begin{array}{r|l}
 & .0\dot{1}5\dot{5}1\dot{8}9\dot{1} \;\big|\; .1246, \text{ nearly.} \\
 & 1 \\
\hline
22 & 55 \\
 & 44 \\
\hline
244 & 1118 \\
 & 976 \\
\hline
2486 & 14291 \\
 & 14916 \\
\hline
\end{array}
$$

CUBE ROOT OF NUMBERS.

(ART. 222, page 176.)

(3.)

$$
\begin{array}{r|l}
 & 94\dot{1}19\dot{2} \;\big|\; 98 \\
 & 729 \\
\hline
\begin{array}{r} 24300 \\ 2160 \\ 64 \\ \hline 26524 \times 8 = \end{array} & \begin{array}{l} 212192 \\ \\ \\ 212192 \\ \hline \end{array}
\end{array}
$$

(4.)

$$
\begin{array}{r|l}
 & 38\dot{9}01\dot{7} \;\big|\; 73 \\
 & 343 \\
\hline
\begin{array}{r} 14700 \\ 630 \\ 9 \\ \hline 15339 \times 3 = \end{array} & \begin{array}{l} 46017 \\ \\ \\ 46017 \\ \hline \end{array}
\end{array}
$$

(5.)

$$
\begin{array}{r|l}
 & 3\dot{7}259\dot{7}0\dot{4} \;\big|\; 334 \\
 & 27 \\
\hline
\begin{array}{r} 2700 \\ 270 \\ 9 \\ \hline 2979 \times 3 = \end{array} & \begin{array}{l} 10259 \\ \\ \\ 8937 \\ \hline \end{array} \\
\begin{array}{r} 326700 \\ 3960 \\ 16 \\ \hline 330676 \times 4 = \end{array} & \begin{array}{l} 1322704 \\ \\ \\ 1322704 \\ \hline \end{array}
\end{array}
$$

(6.)

$$
\begin{array}{rl}
 & 2\,5\,\dot{1}\,2\,3\,\dot{9}\,5\,9\,\dot{1}\ \big|\ 6\,3\,1 \\
 & 2\,1\,6 \\
10800 & \\
540 & \\
9 & \\
\hline
11349 \times 3 = & 35239 \\
 & 34047 \\
\hline
1190700 & \\
1890 & \\
1 & \\
\hline
1192591 \times 1 = & 1192591 \\
 & 1192591 \\
\hline
\end{array}
$$

(7.)

$$
\begin{array}{rl}
 & 4\,\dot{6}\,2\,6\,\dot{8}\,2\,7\,\dot{9}\ \big|\ 3\,\overset{*}{5}\,9 \\
 & 2\,7 \\
2700 & \\
450 & \\
25 & \\
\hline
3175 \times 5 = & 19268 \\
 & 15875 \\
\hline
367500 & \\
9450 & \\
81 & \\
\hline
377031 \times 9 = & 3393279 \\
 & 3393279 \\
\hline
\end{array}
$$

(8.)

$$
\begin{array}{rl}
 & \dot{1}\,4\,8\,\dot{1}.5\,4\,\dot{4}\ \big|\ 11.4 \\
 & 1 \\
300 & \\
30 & \\
1 & \\
\hline
331 \times 1 = & 481 \\
 & 331 \\
\hline
36300 & \\
1320 & \\
16 & \\
\hline
37636 \times 4 = & 150544 \\
 & 150544 \\
\hline
\end{array}
$$

* It is found, by trial, that 7 and 6 are too large, although 27 is contained in 192 seven times.

(9.)

```
                .0 0 8 6 4 9 0 0 0 0 0 0 | .2 0 5 2 +
                     8
 1 2 0 0 0 0              | 6 4 9 0 0 0
     3 0 0 0
         2 5
 1 2 3 0 2 5 × 5 =        | 6 1 5 1 2 5
 1 2 6 0 7 5 0 0          | 3 3 8 7 5 0 0 0
       1 2 3 0 0
               4
 1 2 6 1 9 8 0 4 × 2 =    | 2 5 2 3 9 6 0 8
```

(Art. 223, page 177.)

2. $\sqrt[3]{\frac{125}{343}} = \frac{\sqrt[3]{125}}{\sqrt[3]{343}} = \frac{5}{7}.$

3.

```
           6 8 9 2 1 | 4 1                        5 9 3 1 9 | 3 9
           6 4                                    2 7
 4 8 0 0        | 4 9 2 1           2 7 0 0        | 3 2 3 1 9
   1 2 0                              8 1 0
       1                                8 1
 4 9 2 1 × 1 =  | 4 9 2 1           3 5 9 1 × 9 =  | 3 2 3 1 9
```

$$\sqrt[3]{\frac{68921}{59319}} = \frac{\sqrt[3]{68921}}{\sqrt[3]{59319}} = \frac{41}{39} = 1\frac{2}{39}.$$

(4.)

$\frac{2}{19} = .1\,0\,5\,2\,6\,3\,1\,5\,8$, nearly.

```
                 .1 0 5 2 6 3 1 5 8 | .4 7* 2 +
                  6 4
 4 8 0 0              | 4 1 2 6 3
   8 4 0
     4 9
 5 6 8 9 × 7 =        | 3 9 8 2 3
 6 6 2 7 0 0          | 1 4 4 0 1 5 8
     2 8 2 0
           4
 6 6 5 5 2 4 × 2 =    | 1 3 3 1 0 4 8
```

* It is found, by trial, that 8 is too large, although 48 is contained in 412 eight times.

ROOTS OF MONOMIALS.

(Art. **224**, pp. 178, 179.)

6. Ans. $12\,a^3 b\,c^3$.

It will be a valuable exercise for the student to express the answers in various ways, whenever they are capable of it. Thus,

9. $\pm\dfrac{2\,b\,x^2}{3\,a^2\,y} = \pm\frac{2}{3}\,a^{-2}\,b\,x^2\,y^{-1} = \pm\,2\times 3^{-1}\,a^{-2}\,b\,x^2\,y^{-1}$.

10 $\sqrt[3]{\dfrac{27\,x^3}{729\,y^6}} = \sqrt[3]{\dfrac{x^3}{27\,y^6}} = \dfrac{x}{3\,y^2} = \frac{1}{3}\,x\,y^{-2} = 3^{-1}\,x\,y^{-2}$.

12. $-9\,a^{-1}\,b^{-2} = -\dfrac{9}{a\,b^2}$.

14. We here *divide* each exponent by the *index* of the root, which is 2, or *multiply* each by the *fractional exponent*, which is $\frac{1}{2}$. $\pm\,13\,a^{\frac{n}{2}}\,b^{-\frac{1}{2}}\,c^{-1} = \pm\,\dfrac{13\,a^{\frac{n}{2}}}{b^{\frac{1}{2}}\,c}$.

15. $\dfrac{a\,b^{\frac{1}{7}}}{2\,x^{\frac{4}{7}}\,y} = \frac{1}{2}\,a\,b^{\frac{1}{7}}\,x^{-\frac{4}{7}}\,y^{-1}$.

16. $a\,b^{\frac{m}{n}}\,c^3\,d^{-3} = \dfrac{a\,b^{\frac{m}{n}}\,c^3}{d^3}$.

SQUARE ROOT OF POLYNOMIALS.

(Art. **225**, page 182.)

(4.)

$$\begin{array}{r|l|l}
 & a^4 + 4\,a^2 b + 4\,b^2 & a^2 + 2\,b \\
 & a^4 & \\
\hline
2\,a^2 + 2\,b & 4\,a^2 b + 4\,b^2 & \\
 & 4\,a^2 b + 4\,b^2 & \\
\hline
\end{array}$$

(5.)

$$\begin{array}{r|l}
 & 9x^4 - 12x^3 + 16x^2 - 8x + 4 \quad | \; 3x^2 - 2x + 2 \\
 & 9x^4 \\
\hline
6x^2 - 2x & -12x^3 + 16x^2 \\
 & -12x^3 + 4x^2 \\
\hline
6x^2 - 4x + 2 & 12x^2 - 8x + 4 \\
 & 12x^2 - 8x + 4 \\
\hline
\end{array}$$

(6.)

$$\begin{array}{r|l}
 & x^2 + 4bx + 4b^2 \quad | \; x + 2b \\
 & x^2 \\
\hline
2x + 2b & 4bx + 4b^2 \\
 & 4bx + 4b^2 \\
\hline
\end{array}$$

Ans. $x + 2b$.

(7.)

$$\begin{array}{r|l}
 & a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4 \quad | \; a^2 + 2ab + 3b^2 \\
 & a^4 \\
\hline
2a^2 + 2ab & 4a^3b + 10a^2b^2 \\
 & 4a^3b + 4a^2b^2 \\
\hline
2a^2 + 4ab + 3b^2 & 6a^2b^2 + 12ab^3 + 9b^4 \\
 & 6a^2b^2 + 12ab^3 + 9b^4 \\
\hline
\end{array}$$

(8.)

$$\begin{array}{r|l}
 & a^4 - 2a^3 + 2a^2 - a + \tfrac{1}{4} \quad | \; a^2 - a + \tfrac{1}{2} \\
 & a^4 \\
\hline
2a^2 - a & -2a^3 + 2a^2 \\
 & -2a^3 + a^2 \\
\hline
2a^2 - 2a + \tfrac{1}{2} & a^2 - a + \tfrac{1}{4} \\
 & a^2 - a + \tfrac{1}{4} \\
\hline
\end{array}$$

(9.)

$$\begin{array}{r|l}
 & x^4 - 2x^2 + 1 \quad | \; x^2 - 1 \\
 & x^4 \\
\hline
2x^2 - 1 & -2x^2 + 1 \\
 & -2x^2 + 1 \\
\hline
\end{array}$$

(10.)

$$\begin{array}{r|l}
 & a^2 - 2 + a^{-2} \quad | \; a - a^{-1} \\
 & a^2 \\
\hline
2a - a^{-1} & -2 + a^{-2} \\
 & -2 + a^{-2} \\
\hline
\end{array}$$

(11.)

$$\begin{array}{r|l}
 & 4a^2 - 12ab + 9b^2 + 4ax - 6bx + x^2 \;\big|\; 2a - 3b + x \\
 & 4a^2 \\ \hline
4a - 3b & -12ab + 9b^2 \\
 & -12ab + 9b^2 \\ \hline
4a - 6b + x & 4ax - 6bx + x^2 \\
 & 4ax - 6bx + x^2 \\ \hline
\end{array}$$

(12.)

$$\begin{array}{r|l}
 & a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4 \;\big|\; a^2 + 4ab + 4b^2 \\
 & a^4 \\ \hline
2a^2 + 4ab & 8a^3b + 24a^2b^2 \\
 & 8a^3b + 16a^2b^2 \\ \hline
2a^2 + 8ab + 4b^2 & 8a^2b^2 + 32ab^3 + 16b^4 \\
 & 8a^2b^2 + 32ab^3 + 16b^4 \\ \hline
\end{array}$$

$$\begin{array}{r|l}
 & a^2 + 4ab + 4b^2 \;\big|\; a + 2b \\
 & a^2 \\ \hline
2a + 2b & 4ab + 4b^2 \\
 & 4ab + 4b^2 \\ \hline
\end{array}$$

CUBE ROOT OF POLYNOMIALS.

(Art. 226, pp. 185, 186.)

(4.)

$$\begin{array}{r|l}
 & x^3 + 3x^2y + 3xy^2 + y^3 \;\big|\; x + y \\
 & x^3 \\ \hline
3x^2 + 3xy + y^2 & 3x^2y + 3xy^2 + y^3 \\
 & 3x^2y + 3xy^2 + y^3 \\ \hline
\end{array}$$

(5.)

$$\begin{array}{r|l}
 & y^6 - 3y^5 + 5y^3 - 3y - 1 \;\big|\; y^2 - y - 1 \\
 & y^6 \\ \hline
3y^4 - 3y^3 + y^2 & -3y^5 + 5y^3 \\
 & -3y^5 + 3y^4 - y^3 \\ \hline
3y^4 - 6y^3 + 3y + 1 & -3y^4 + 6y^3 - 3y - 1 \\
 & -3y^4 + 6y^3 - 3y - 1 \\ \hline
\end{array}$$

Note. In the last complete divisor, $3y^2$ and $-3y^2$ cancel each other.

(6.)

$$\begin{array}{r|l|l} & 27\,x^3 + 54\,x^2 y + 36\,x y^2 + 8\,y^3 & 3\,x + 2\,y \\ & 27\,x^3 & \\ \hline 27\,x^2 + 18\,x y + 4\,y^2 & 54\,x^2 y + 36\,x y^2 + 8\,y^3 & \\ & 54\,x^2 y + 36\,x y^2 + 8\,y^3 & \\ \hline \end{array}$$

Ans. $3\,x + 2\,y$.

7. It will be observed that the solution of this example is precisely the same as that of the 3d, except that m is substituted for x.

(8.)

$$\begin{array}{r|l|l} & a^3 + 3\,a + 3\,a^{-1} + a^{-3} & a + a^{-1} \\ & a^3 & \\ \hline 3\,a^2 + 3 + a^{-2} & 3\,a + 3\,a^{-1} + a^{-3} & \\ & 3\,a + 3\,a^{-1} + a^{-3} & \\ \hline \end{array}$$

RADICALS.

REDUCTION OF RADICALS.

(ART. 233, page 188.)

2. $\sqrt{9\,a^4\,x} = \sqrt{9\,a^4 \times x} = \sqrt{9\,a^4}\,\sqrt{x} = 3\,a^2\,\sqrt{x}$.

3. $\sqrt{32\,a^2\,x} = \sqrt{16\,a^2 \times 2\,x} = \sqrt{16\,a^2}\,\sqrt{2\,x} = 4\,a\,\sqrt{2\,x}$.

4. $7\,\sqrt{80\,x} = 7\,\sqrt{16 \times 5\,x} = 7 \times 4\,\sqrt{5\,x} = 28\,\sqrt{5\,x}$.

5. $a\sqrt{125\,b^3} = a\sqrt{25\,b^2 \times 5\,b} = a \times 5\,b\,\sqrt{5\,b} = 5\,a\,b\,\sqrt{5\,b}$.

6. $\sqrt[3]{64\,a^5\,b^6} = \sqrt[3]{64\,a^3\,b^6 \times a^2} = 4\,a\,b^2\,\sqrt[3]{a^2}$.

7. $\sqrt{50\,a\,b^2\,c^2} = \sqrt{25\,b^2\,c^2 \times 2\,a} = 5\,b\,c\,\sqrt{2\,a}$, Ans.

8. $(a\,x^3 + b\,x^4)^{\frac{1}{3}} = \sqrt[3]{x^3\,(a + b\,x)} = x\,(a + b\,x)^{\frac{1}{3}}$.

9. $2\,(x^3 - a^2\,x^2)^{\frac{1}{2}} = 2\,\sqrt{x^2\,(x - a^2)} = 2\,x\,(x - a^2)^{\frac{1}{2}}$.

10. $\sqrt[3]{5\,(a^3 + a^4\,b)} = \sqrt[3]{5\,a^3\,(1 + a\,b)} = a\,\sqrt[3]{5\,(1 + a\,b)}$.

11. $6\sqrt{54\,a^3b^3c} = 6\sqrt{9a^2b^2 \times 6\,abc} = 6 \times 3\,ab\sqrt{6\,abc}$
$= 18\,ab\sqrt{6\,abc}.$

12. $3\sqrt[5]{32\,a^6\,b^2\,c^5} = 3\sqrt[5]{32\,a^5\,c^5 \times a\,b^2} = 3 \times 2\,ac\sqrt[5]{a\,b^2}$
$= 6\,ac\sqrt[5]{a\,b^2}.$

13. $(72\,x + 108\,y)^{\frac{1}{2}} = \sqrt{36\,(2\,x + 3\,y)} = 6\,(2\,x + 3\,y)^{\frac{1}{2}}.$

14. $5\,(a - b)\sqrt{a^2c + 2\,abc + b^2c} = 5\,(a - b)\sqrt{c\,(a + b)^2}$
$= 5\,(a - b)\,(a + b)\sqrt{c} = 5\,(a^2 - b^2)\sqrt{c}.$

(Art. **234**, p. 183.)

3. $2\sqrt{\frac{3}{8}} = 2\sqrt{\frac{3}{8} \times \frac{2}{2}} = 2\sqrt{\frac{6}{16}} = 2\sqrt{\frac{1}{16} \times 6} = 2 \times \frac{1}{4}\sqrt{6}$
$= \frac{2}{4}\sqrt{6} = \frac{1}{2}\sqrt{6}.$

4. $\frac{3}{4}\sqrt{\frac{2\,a\,x^3}{3}} = \frac{3}{4}\sqrt{\frac{2\,a\,x^3}{3} \times \frac{3}{3}} = \frac{3}{4}\sqrt{\frac{6\,a\,x^3}{9}} = \frac{3}{4}\sqrt{\frac{x^2}{9} \times 6\,ax}$
$= \frac{3}{4} \times \frac{x}{3}\sqrt{6\,ax} = \frac{x}{4}\sqrt{6\,ax}.$

5. $\left(\frac{a^4\,b^3}{4}\right)^{\frac{1}{3}} = \left(\frac{a^4\,b^3}{4} \times \frac{2}{2}\right)^{\frac{1}{3}} = \left(\frac{2\,a^4\,b^3}{8}\right)^{\frac{1}{3}} = \left(\frac{a^3\,b^3}{8} \times 2\,a\right)^{\frac{1}{3}}$
$= \frac{ab}{2}\,(2\,a)^{\frac{1}{3}}.$

6. $4\sqrt{\frac{3\,a^3}{8\,b}} = 4\sqrt{\frac{3\,a^3}{8\,b} \times \frac{2\,b}{2\,b}} = 4\sqrt{\frac{6\,a^3\,b}{16\,b^2}} = 4\sqrt{\frac{a^2}{16\,b^2} \times 6\,ab}$
$= 4 \times \frac{a}{4\,b}\sqrt{6\,ab} = \frac{a}{b}\sqrt{6\,ab}.$

7. $2\left(\frac{5\,a^5\,b}{8\,c^3}\right)^{\frac{1}{4}} = 2\left(\frac{5\,a^5\,b}{8\,c^3} \times \frac{2\,c}{2\,c}\right)^{\frac{1}{4}} = 2\left(\frac{10\,a^5\,b\,c}{16\,c^4}\right)^{\frac{1}{4}}$
$= 2\left(\frac{a^4}{16\,c^4} \times 10\,abc\right)^{\frac{1}{4}} = 2 \times \frac{a}{2\,c}\left(10\,abc\right)^{\frac{1}{4}}$
$= \frac{a}{c}\left(10\,abc\right)^{\frac{1}{4}}.$

(Art. 235, page 190.)

2. $3ax = (3ax)^{\frac{2}{2}} = \sqrt{(3ax)^2} = \sqrt{9a^2x^2}$.

3. $-5a^2b = \sqrt[3]{(-5a^2b)^3} = \sqrt[3]{-125a^6b^3}$.

4. $2x-3 = \left((2x-3)^2\right)^{\frac{1}{2}} = (4x^2-12x+9)^{\frac{1}{2}}$.

5. $2a^2x^3y = \sqrt[4]{(2a^2x^3y)^4} = \sqrt[4]{16x^8x^{12}y^4}$, Ans.

6. $\dfrac{3ax^2}{2by^3} = \sqrt[5]{\dfrac{(3ax^2)^5}{(2by^3)^5}} = \sqrt[5]{\dfrac{243a^5x^{10}}{32b^5y^{15}}}$.

(Art. 236, page 191.)

2. $5\sqrt[3]{xy-1} = \sqrt[3]{5^3(xy-1)} = \sqrt[3]{125xy-125}$.

3. $2a^2\sqrt[4]{3ab} = a^2\sqrt[4]{2^4 \times 3ab} = a^2\sqrt[4]{48ab}$.

4. $(a+b)\sqrt{c} = \sqrt{c(a+b)^2} = \sqrt{c(a^2+2ab+b^2)}$
$= (a^2c + 2abc + b^2c)^{\frac{1}{2}}$.

5. $\dfrac{a}{b}\sqrt{\dfrac{b^2c}{a^2-b^2}} = \sqrt{\dfrac{a^2}{b^2}\left(\dfrac{b^2c}{a^2-b^2}\right)} = \sqrt{\dfrac{a^2b^2c}{a^2b^2-b^4}}$.

(Art. 237, page 192.)

3. $3\sqrt[4]{3} = 3(3)^{\frac{1}{4}} = 3(3)^{\frac{3}{12}} = 3\sqrt[12]{3^3} = 3\sqrt[12]{27}$.

4. $\begin{cases} \sqrt[4]{a} = a^{\frac{1}{4}} = a^{\frac{3}{12}} = (a^3)^{\frac{1}{12}}. \\ (5b)^{\frac{1}{6}} = (5b)^{\frac{2}{12}} = [(5b)^2]^{\frac{1}{12}} = (25b^2)^{\frac{1}{12}}. \\ (a^2+b^2)^{\frac{1}{3}} = (a^2+b^2)^{\frac{4}{12}} = [(a^2+b^2)^4]^{\frac{1}{12}}. \end{cases}$

5. $\begin{cases} \sqrt{a} = a^{\frac{1}{2}} = a^{\frac{15}{30}} = \sqrt[30]{a^{15}}. \\ \sqrt[3]{a-b} = (a-b)^{\frac{1}{3}} = (a-b)^{\frac{10}{30}} = \sqrt[30]{(a-b)^{10}} \\ \sqrt[5]{a+b} = (a+b)^{\frac{1}{5}} = (a+b)^{\frac{6}{30}} = \sqrt[30]{(a+b)^6}. \end{cases}$

ADDITION OF RADICALS.

(Art. **238**, pp. 193, 194.)

3. $5\sqrt{98x} = 5\sqrt{49 \times 2x} = 5 \times 7\sqrt{2x} = 35\sqrt{2x}$;
$35\sqrt{2x} + 10\sqrt{2x} = 45\sqrt{2x}$.

4. $\sqrt[3]{48a} = \sqrt[3]{8 \times 6a} = 2\sqrt[3]{6a}$
$\sqrt[3]{162a} = \sqrt[3]{27 \times 6a} = 3\sqrt[3]{6a}$
$\text{Sum} = 5\sqrt[3]{6a}$

5. $\sqrt[4]{32} = \sqrt[4]{16 \times 2} = 2\sqrt[4]{2}$; $2\sqrt[4]{2} + 5\sqrt[4]{2} = 7\sqrt[4]{2}$.

6. $\sqrt{3a^2b} = \sqrt{a^2 \times 3b} = a\sqrt{3b}$
$\sqrt{3x^2b} = \sqrt{x^2 \times 3b} = x\sqrt{3b}$
$\text{Sum} = (a + x)\sqrt{3b}$

7. $5\sqrt{20a^2x} = 5\sqrt{4a^2 \times 5x} = 5 \times 2a\sqrt{5x} = 10a\sqrt{5x}$
$3\sqrt{45a^2x} = 3\sqrt{9a^2 \times 5x} = 3 \times 3a\sqrt{5x} = 9a\sqrt{5x}$
$\text{Sum} = 19a\sqrt{5x}$

8. $(3a^2b)^{\frac{1}{2}} = (a^2 \times 3b)^{\frac{1}{2}} = a(3b)^{\frac{1}{2}}$
$(27a^2b)^{\frac{1}{2}} = (9a^2 \times 3b)^{\frac{1}{2}} = 3a(3b)^{\frac{1}{2}}$
$\text{Sum} = 4a(3b)^{\frac{1}{2}}$

9. $(45c^3)^{\frac{1}{2}} = (9c^2 \times 5c)^{\frac{1}{2}} = 3c(5c)^{\frac{1}{2}}$
$(80c^3)^{\frac{1}{2}} = (16c^2 \times 5c)^{\frac{1}{2}} = 4c(5c)^{\frac{1}{2}}$
$(5a^2c)^{\frac{1}{2}} = (a^2 \times 5c)^{\frac{1}{2}} = a(5c)^{\frac{1}{2}}$
$\text{Sum} = (7c + a)(5c)^{\frac{1}{2}}$, Ans.

10. $\sqrt[3]{b^4y} = \sqrt[3]{b^3 \times by} = b\sqrt[3]{by}$
$\sqrt[3]{by^4} = \sqrt[3]{y^3 \times by} = y\sqrt[3]{by}$
$\text{Sum} = (b + y)\sqrt[3]{by}$

12. $12\sqrt[3]{\tfrac{1}{4}}=12\sqrt[3]{\tfrac{2}{8}}=12\sqrt[3]{\tfrac{1}{8}\times 2}=12\times\tfrac{1}{2}\sqrt[3]{2}=6\sqrt[3]{2}$

$3\sqrt[3]{\tfrac{1}{32}}=3\sqrt[3]{\tfrac{2}{64}}=3\sqrt[3]{\tfrac{1}{64}\times 2}=\ 3\times\tfrac{1}{4}\sqrt[3]{2}=\tfrac{3}{4}\sqrt[3]{2}$

Sum $=\tfrac{27}{4}\sqrt[3]{2}$

13. $\sqrt{4x}=2\sqrt{x}$. The radicals being all dissimilar, their addition can only be *indicated* by the sign $+$.

14. $\sqrt{4x+4}=\sqrt{4(x+1)}=2\sqrt{x+1}$;

$$\sqrt{x+1}+2\sqrt{x+1}=3\sqrt{x+1}.$$

15. $(x^3y)^{\frac{1}{3}}=xy^{\frac{1}{3}}$; $(xy^3)^{\frac{1}{3}}=x^{\frac{1}{3}}y$; $(8a^4y)^{\frac{1}{3}}=(8a^3\times ay)^{\frac{1}{3}}$ $=2a(ay)^{\frac{1}{3}}$. These radicals are all dissimilar.

16. $\frac{1}{2}\sqrt{a^2bc}=\frac{1}{2}\times a\sqrt{bc}=\frac{a}{2}\sqrt{bc}$

$\frac{1}{3}\sqrt{4bcx^4}=\frac{1}{3}\times 2x^2\sqrt{bc}=\frac{2x^2}{3}\sqrt{bc}$

$$\left(\frac{a}{2}+\frac{2x^2}{3}\right)\sqrt{bc}$$

17. $\sqrt[3]{16a^3b}=\sqrt[3]{8a^3\times 2b}=2a\sqrt[3]{2b}$

$\sqrt[3]{54a^3b}=\sqrt[3]{27a^3\times 2b}=3a\sqrt[3]{2b}$

$$5a\sqrt[3]{2b}$$

$\sqrt{4a^2b}=2a\sqrt{b}$; $\sqrt{a^2b}=a\sqrt{b}$; $2a\sqrt{b}+a\sqrt{b}=3a\sqrt{b}$

It will be seen that $5a\sqrt[3]{2b}$ and $3a\sqrt{b}$ are dissimilar, because the radicals are of a different degree, while the quantities under the radical sign are also different. Their addition can, therefore, be only *indicated.*

SUBTRACTION OF RADICALS.

(Art. **239**, page 195.)

2. $\sqrt{45a}=\sqrt{9\times 5a}=3\sqrt{5a}$; $3\sqrt{5a}-\sqrt{5a}=2\sqrt{5a}$

3. $\sqrt[3]{192}=\sqrt[3]{64\times 3}=4\sqrt[3]{3}$

$\sqrt[3]{24}=\sqrt[3]{8\times 3}=2\sqrt[3]{3}$

Difference $=2\sqrt[3]{3}$

4. $(9a^4x)^{\frac{1}{2}} = 3a^2x^{\frac{1}{2}}$; $(4a^4x)^{\frac{1}{2}} = 2a^2x^{\frac{1}{2}}$; $3a^2x^{\frac{1}{2}} - 2a^2x^{\frac{1}{2}}$
$= a^2x^{\frac{1}{2}} = a^2\sqrt{x}$.

5.
$$\begin{aligned} b\sqrt[3]{8a^6b} &= b\sqrt[3]{8a^6 \times b} = 2a^2b\sqrt[3]{b} \\ a\sqrt[3]{a^3b^4} &= a\sqrt[3]{a^3b^3 \times b} = a^2b\sqrt[3]{b} \\ \text{Difference} &= a^2b\sqrt[3]{b} \end{aligned}$$

6. The radicals are already similar; hence we merely take the difference of the coefficients 4 and 3, and annex the common radical.

7.
$$\begin{aligned} \sqrt{108ax^2} &= \sqrt{36x^2 \times 3a} = 6x\sqrt{3a} \\ \sqrt{48ax^2} &= \sqrt{16x^2 \times 3a} = 4x\sqrt{3a} \\ \text{Difference} &= 2x\sqrt{3x}, \text{ Ans.} \end{aligned}$$

8.
$$\begin{aligned} \sqrt{\tfrac{3}{4}} &= \sqrt{\tfrac{1}{4} \times 3} = \tfrac{1}{2}\sqrt{3} = \tfrac{3}{6}\sqrt{3} \\ \sqrt{\tfrac{1}{3}} = \sqrt{\tfrac{1}{3} \times \tfrac{3}{3}} = \sqrt{\tfrac{3}{9}} &= \sqrt{\tfrac{1}{9} \times 3} = \tfrac{1}{3}\sqrt{3} = \tfrac{2}{6}\sqrt{3} \\ \text{Difference} &= \tfrac{1}{6}\sqrt{3} \end{aligned}$$

9.
$$\begin{aligned} 2\sqrt{3a^2b^2c} &= 2\sqrt{a^2b^2 \times 3c} = 2ab\sqrt{3c} \\ \sqrt{5ab^3} &= \sqrt{b^2 \times 5ab} = b\sqrt{5ab} \\ \text{Difference} &= 2ab\sqrt{3c} - b\sqrt{5ab} \end{aligned}$$

10.
$$\begin{aligned} \sqrt[4]{32a} &= \sqrt[4]{16 \times 2a} = 2\sqrt[4]{2a} \\ 2\sqrt[3]{40a} &= 2\sqrt[3]{8 \times 5a} = 4\sqrt[3]{5a} \\ \text{Difference} &= 2\sqrt[4]{2a} - 4\sqrt[3]{5a} = 2(\sqrt[4]{2a} - 2\sqrt[3]{5a}) \end{aligned}$$

11.
$$\begin{aligned} \sqrt[3]{8a^3b + 16a^4} &= \sqrt[3]{8a^3(b + 2a)} = 2a\sqrt[3]{2a + b} \\ \sqrt[3]{b^4 + 2ab^3} &= \sqrt[3]{b^3(b + 2a)} = b\sqrt[3]{2a + b} \\ \text{Difference} &= (2a - b)\sqrt[3]{2a + b} \end{aligned}$$

MULTIPLICATION OF RADICALS.

(ART. 240, page 197.)

4. $6\sqrt{54} \times 3\sqrt{2} = 6 \times 3\sqrt{54 \times 2} = 18\sqrt{108}$
$= 18\sqrt{36 \times 3} = 18 \times 6\sqrt{3} = 108\sqrt{3}$, Ans.

5. $7\sqrt{axy} \times 3\sqrt{2ax} = 21\sqrt{2a^2x^2y} = 21ax\sqrt{2y}$.

6. $a\sqrt[m]{x} \times b\sqrt[n]{y} = a\sqrt[mn]{x^n} \times b\sqrt[mn]{y^m} = ab\sqrt[mn]{x^n y^m}$.

7. $4\sqrt[3]{ax} \times 3\sqrt{xy} = 4\sqrt[6]{a^2x^2} \times 3\sqrt[6]{x^3y^3} = 12\sqrt[6]{a^2x^5y^3}$.

8. $\frac{1}{4}\sqrt{6} \times \frac{2}{15}\sqrt{9} = \frac{1}{4} \times \frac{2}{15}\sqrt{6 \times 9} = \frac{1}{4} \times \frac{2}{15} \times 3\sqrt{6}$
$= \frac{1}{10}\sqrt{6}$.

NOTE. The second of the given radicals, $\frac{2}{15}\sqrt{9}$, is rational, and reduces to $\frac{2}{15} \times 3 = \frac{2}{5}$. Multiplying $\frac{1}{4}\sqrt{6}$ by $\frac{2}{5}$ we obtain $\frac{1}{10}\sqrt{6}$, the same result as before.

9. $2\sqrt[3]{\frac{2}{3}} \times 3\sqrt[3]{\frac{5}{6}} = 2 \times 3\sqrt[3]{\frac{2}{3} \times \frac{5}{6}} = 6\sqrt[3]{\frac{5}{9}} = 6\sqrt[3]{\frac{5}{9} \times \frac{3}{3}}$
$= 6\sqrt[3]{\frac{15}{27}} = 6\sqrt[3]{\frac{1}{27} \times 15} = \frac{6}{3}\sqrt[3]{15} = 2\sqrt[3]{15}$.

10. $3b^{\frac{1}{3}} \times 4a^{\frac{1}{4}} = 12a^{\frac{1}{4}}b^{\frac{1}{3}} = 12a^{\frac{3}{12}}b^{\frac{4}{12}} = 12\sqrt[12]{a^3b^4}$.

11. $3a\sqrt[4]{8a^2} \times 2b\sqrt[4]{4a^2c} = 3a \times 2b\sqrt[4]{8a^2 \times 4a^2c}$
$= 6ab\sqrt[4]{32a^4c} = 6ab\sqrt[4]{16a^4 \times 2c} = 6ab \times 2a\sqrt[4]{2c}$
$= 12a^2b\sqrt[4]{2c}$.

12. $(a+b)^{\frac{1}{2}} \times (a+b)^{\frac{2}{3}} = (a+b)^{\frac{3}{6}} \times (a+b)^{\frac{4}{6}} = (a+b)^{\frac{7}{6}}$.

13. $\sqrt{\frac{2ax}{3c}} \times \sqrt{\frac{9ab}{2x}} = \sqrt{\frac{2ax}{3c} \times \frac{9ab}{2x}} = \sqrt{\frac{18a^2b}{6c}}$
$= \sqrt{\frac{3a^2b}{c}}$.

14. $\sqrt[3]{6a^{\frac{1}{2}}bc^{-1}} \times \sqrt[3]{3^{-1}a^{-\frac{1}{2}}bc^2} = \sqrt[3]{6a^{\frac{1}{2}}bc^{-1} \times 3^{-1}a^{-\frac{1}{2}}bc^2}$
$= \sqrt[3]{\frac{6}{3}b^2c} = \sqrt[3]{2b^2c}$.

(Art. **241**, page 198.)

(2.)

$$\begin{array}{l} 4+2\sqrt{2} \\ \underline{2-\sqrt{2}\qquad\quad} \\ 8+4\sqrt{2} \\ \quad\;\; -4\sqrt{2}-4 \\ \hline 8 \qquad\qquad -4=4 \end{array}$$

3. $(a+x)^{\frac{2}{3}} \times (a-x)^{\frac{2}{3}} = [(a+x)\,(a-x)]^{\frac{2}{3}} = (a^2-x^2)^{\frac{2}{3}}$.

4. $\sqrt{a+b} \times \sqrt{a+b} = \sqrt{(a+b)^2} = a+b$.

Note. It will be seen that multiplying the square root of any quantity by itself removes the radical.

(5.)

$$\begin{array}{l} \frac{1}{2}+\frac{1}{2}\sqrt{5} \\ \underline{\frac{3}{4}+\frac{1}{4}\sqrt{5}} \\ \frac{3}{8}+\frac{3}{8}\sqrt{5} \\ \quad\;\; \frac{1}{8}\sqrt{5}+\frac{5}{8} \\ \hline \frac{3}{8}+\frac{4}{8}\sqrt{5}+\frac{5}{8}=1+\frac{1}{2}\sqrt{5} \end{array}$$

(6.)

$$\begin{array}{l} \sqrt{x}+\sqrt{a+x} \\ \underline{\sqrt{a+x}\qquad\qquad\quad} \\ \sqrt{ax+x^2}+a+x \end{array}$$

(7.)

$$\begin{array}{l} a^{\frac{3}{2}}+b^{\frac{2}{3}} \\ \underline{a^{\frac{1}{2}}-2b^{\frac{3}{2}}} \\ a^{\frac{4}{2}}+a^{\frac{1}{2}}b^{\frac{2}{3}} \\ \qquad\qquad -2a^{\frac{3}{2}}b^{\frac{3}{2}}-2b^{\frac{13}{6}} \\ \hline a^2+a^{\frac{1}{2}}b^{\frac{2}{3}}-2a^{\frac{3}{2}}b^{\frac{3}{2}}-2b^{\frac{13}{6}} \end{array}$$

DIVISION OF RADICALS.

(Art. **242**, pp. 200, 201.)

3. $\dfrac{\sqrt{40}}{\sqrt{2}} = \sqrt{\dfrac{40}{2}} = \sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$.

4. $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\dfrac{a}{b}}\,.$

5. $\dfrac{\sqrt[3]{135}}{\sqrt[3]{5}} = \sqrt[3]{\dfrac{135}{5}} = \sqrt[3]{27} = 3.$ Ans. 3.

Note. The same result can be obtained by reducing $\sqrt[3]{135}$ to its simplest form, $3\sqrt[3]{5}$, and then dividing by $\sqrt[3]{5}$.

6. $\dfrac{4\sqrt{a^3}}{2\sqrt[4]{a}} = \dfrac{4\sqrt[4]{a^6}}{2\sqrt[4]{a}} = \dfrac{4}{2}\sqrt[4]{\dfrac{a^6}{a}} = 2\sqrt[4]{a^5} = 2\,a\sqrt[4]{a}.$

7. $\dfrac{4\sqrt[3]{a\,x}}{3\sqrt{x\,y}} = \dfrac{4\sqrt[6]{a^2x^2}}{3\sqrt[6]{x^3y^3}} = \dfrac{4}{3}\sqrt[6]{\dfrac{a^2x^2}{x^3y^3}} = \dfrac{4}{3}\sqrt[6]{\dfrac{a^2}{x\,y^3}}.$

8. $\dfrac{b\,c\sqrt[n]{a\,b}}{b\sqrt[n]{a}} = \dfrac{b\,c}{b}\sqrt[n]{\dfrac{a\,b}{a}} = c\sqrt[n]{b}.$

9. $\dfrac{(1-x^2)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} = \left(\dfrac{1-x^2}{1+x}\right)^{\frac{1}{2}} = (1-x)^{\frac{1}{2}}.$

10. $\sqrt{\dfrac{a}{b}} \div \sqrt{\dfrac{c}{d}} = \sqrt{\dfrac{a}{b} \div \dfrac{c}{d}} = \sqrt{\dfrac{a}{b} \times \dfrac{d}{c}} = \sqrt{\dfrac{a\,d}{b\,c}}\,.$

11. $\frac{1}{2}\sqrt[3]{\frac{1}{2}} \div \frac{1}{3}\sqrt[3]{\frac{1}{3}} = (\frac{1}{2} \div \frac{1}{3})\sqrt[3]{\frac{1}{2} \div \frac{1}{3}} = (\frac{1}{2} \times \frac{3}{1})\sqrt[3]{\frac{1}{2} \times \frac{3}{1}}$
$= \frac{3}{2}\sqrt[3]{\frac{3}{2}} = \frac{3}{2}\sqrt[3]{\frac{3}{2} \times \frac{4}{4}} = \frac{3}{2}\sqrt[3]{\frac{12}{8}} = \frac{3}{2} \times \frac{1}{2}\sqrt[3]{12}$
$= \frac{3}{4}\sqrt[3]{12}.$

12. $\dfrac{\sqrt{72} + \sqrt{32-4}}{\sqrt{8}} = \sqrt{\dfrac{72}{8}} + \sqrt{\dfrac{28}{8}} = \sqrt{9} + \sqrt{\dfrac{7}{2}}$

$= 3 + \sqrt{\dfrac{7}{2} \times \dfrac{2}{2}} = 3 + \sqrt{\dfrac{14}{4}} = 3 + \frac{1}{2}\sqrt{14}\,.$

13. The radicals, being alike, cancel each other in dividing, and there is nothing left but the coefficients m and n, which give $\frac{m}{n}$ as the result.

14. $\dfrac{\sqrt{a^2 - b^2}}{a - b} = \dfrac{\sqrt{(a+b)(a-b)}}{\sqrt{(a-b)^2}} = \sqrt{\dfrac{(a+b)(a-b)}{(a-b)^2}}$

$= \sqrt{\dfrac{a+b}{a-b}}\,.$

(15.)

$$\begin{array}{r} a - 2b^{\frac{1}{2}})\ a^2 + ab^{\frac{1}{2}} - 6b\ (a + 3b^{\frac{1}{2}} \\ \underline{a^2 - 2ab^{\frac{1}{2}}} \qquad\qquad \\ 3ab^{\frac{1}{2}} - 6b \\ \underline{3ab^{\frac{1}{2}} - 6b} \end{array}$$

(16.)

$$\begin{array}{r} a^{\frac{1}{2}} - 4b^{\frac{1}{2}})\ a^2 + 2a^{\frac{1}{2}}b^{\frac{2}{3}} - 4a^{\frac{3}{2}}b^{\frac{1}{2}} - 8b^{\frac{7}{6}}\ (a^{\frac{3}{2}} + 2b^{\frac{2}{3}} \\ \underline{a^2 - 4a^{\frac{3}{2}}b^{\frac{1}{2}}} \qquad\qquad\qquad\qquad \\ 2a^{\frac{1}{2}}b^{\frac{2}{3}} - 8b^{\frac{7}{6}} \\ \underline{2a^{\frac{1}{2}}b^{\frac{2}{3}} - 8b^{\frac{7}{6}}} \end{array}$$

INVOLUTION OF RADICALS.

(ART. 243, page 202.)

3. $(5a^{\frac{1}{3}})^2 = 5^2a^{\frac{2}{3}} = 25a^{\frac{2}{3}}$.

4. $(5a\sqrt[3]{y})^3 = 5^3a^3\sqrt[3]{y^3} = 125a^3y$.

5. $(x^2\sqrt[3]{6})^2 = (x^2)^2\sqrt[3]{6^2} = x^4\sqrt[3]{36}$.

6. $(4a^2\sqrt{3c})^4 = 4^4a^8\sqrt{3^4c^4} = 4^4a^8 \times 3^2c^2 = 2304a^8c^2$, Ans.

7. $(3\sqrt[4]{25ax})^2 = 3^2\sqrt{25ax} = 9 \times 5\sqrt{ax} = 45\sqrt{ax}$.

8.
$$\begin{array}{l} \sqrt{3} - \sqrt{2} \\ \underline{\sqrt{3} - \sqrt{2}} \\ 3 - \sqrt{6} \\ \underline{\quad - \sqrt{6} + 2} \\ 3 - 2\sqrt{6} + 2 = 5 - 2\sqrt{6} \\ 5 - 2\sqrt{6} \\ \underline{5 - 2\sqrt{6}} \\ 25 - 10\sqrt{6} \\ \underline{\quad - 10\sqrt{6} + 24} \\ 25 - 20\sqrt{6} + 24 = 49 - 20\sqrt{6} \end{array}$$

9. $(\sqrt[m]{2a})^n = \sqrt[m]{2^n a^n}$.

10. $(\sqrt{3} + x\sqrt{3})^2 = \sqrt{3^2} + 2x\sqrt{3^2} + x^2\sqrt{3^2}$
$= 3 + 6x + 3x^2$.

11. $(x^{\frac{1}{2}} - y^{-\frac{2}{3}})^2 = (x^{\frac{1}{2}})^2 - 2x^{\frac{1}{2}}y^{-\frac{2}{3}} + (y^{-\frac{2}{3}})^2$
$= x - 2x^{\frac{1}{2}}y^{-\frac{2}{3}} + y^{-\frac{4}{3}}$.

EVOLUTION OF RADICALS.

(Art. 244, page 204.)

3. $(\sqrt{ab^2})^{\frac{1}{3}} = \sqrt[6]{ab^2}$.

4. $\sqrt{4\sqrt[3]{5c}} = \sqrt{4} \times \sqrt[6]{5c} = 2\sqrt[6]{5c}$.

5. $\sqrt[3]{a^{-1}(xy)^{\frac{1}{2}}} = a^{-\frac{1}{3}}(xy)^{\frac{1}{6}} = a^{-\frac{1}{3}}x^{\frac{1}{6}}y^{\frac{1}{6}}$.

6. $\sqrt{25\sqrt[3]{4a^2b}} = \sqrt{25} \times \sqrt[6]{4a^2b} = 5\sqrt[6]{4a^2b}$, Ans.

7. $a\sqrt{a} = \sqrt{a^2 \times a} = \sqrt{a^3}$; $(\sqrt{a^3})^{\frac{1}{6}} = \sqrt[12]{a^3} = \sqrt[4]{a}$.

8. $\frac{a}{3}\sqrt{\frac{a}{3}} = \sqrt{\frac{a^2}{3^2} \times \frac{a}{3}} = \sqrt{\frac{a^3}{3^3}}$; $\left(\sqrt{\frac{a^3}{3^3}}\right)^{\frac{1}{3}} = \sqrt{\frac{a}{3}} = \sqrt{\frac{a}{3} \times \frac{3}{3}}$
$= \sqrt{\frac{3a}{9}} = \frac{1}{3}\sqrt{3a}$.

9. $\sqrt[3]{125x^{\frac{1}{2}}} = 5x^{\frac{1}{6}}$.

10. $\sqrt[4]{64a^8b^4\sqrt{2cd}} = \sqrt[4]{16a^8b^4\sqrt{4^2 \times 2cd}} = 2a^2b\sqrt[8]{32cd}$.

(11.)

$$\begin{array}{r|l|l}
 & x^{\frac{4}{3}} + 6x^{\frac{2}{3}}y^{\frac{1}{2}} + 9y & x^{\frac{2}{3}} + 3y^{\frac{1}{2}} \\
 & x^{\frac{4}{3}} & \\
\hline
2x^{\frac{2}{3}} + 3y^{\frac{1}{2}} & 6x^{\frac{2}{3}}y^{\frac{1}{2}} + 9y & \\
 & 6x^{\frac{2}{3}}y^{\frac{1}{2}} + 9y & \\
\hline
\end{array}$$

RATIONALIZATION.

(Art. 246, page 205.)

3. The factor $x^{\frac{1}{4}}$ will rationalize $x^{\frac{3}{4}}$, because $x^{\frac{3}{4}} \times x^{\frac{1}{4}} = x^{\frac{4}{4}} = x$.

4. $4\sqrt[3]{a\,b^2} = 4\,a^{\frac{1}{3}}\,b^{\frac{2}{3}}$; $\quad 4\,a^{\frac{1}{3}}\,b^{\frac{2}{3}} \times a^{\frac{2}{3}}\,b^{\frac{1}{3}} = 4\,a\,b$;
$a^{\frac{2}{3}}\,b^{\frac{1}{3}} = \sqrt[3]{a^2\,b}$.

Note. It is evident, without the use of fractional exponents, that $4\sqrt[3]{a\,b^2} \times \sqrt[3]{a^2\,b} = 4\sqrt[3]{a^3\,b^3} = 4\,a\,b$. In general, such a radical quantity is rationalized by means of another radical of the same degree, with such exponents as will make the product a perfect power of the same degree as the radical, and thus remove the radical sign.

5. $a^{-\frac{1}{2}} \times a^{\frac{3}{2}} = a^{\frac{2}{2}} = a$. The factor $a^{-\frac{1}{2}}$ will rationalize $a^{-\frac{1}{2}}$, as the product will be $a^{-1} = \frac{1}{a}$; but it will not make the exponent positive, that is, it will not make the product integral.

Note. The rule, as stated in the Algebra, is intended to apply to radicals in their simplest form. (Art. 233.) Thus, to rationalize $\sqrt{a^3}$, we first reduce it to $a\sqrt{a}$, and then multiply by such a factor as will rationalize $\sqrt{a}$, without reference to the coefficient a, that being already rational. If, however, we are required to rationalize $\sqrt{a^3}$, or $a^{\frac{3}{2}}$, as it stands, we must *multiply the given surd by the same quantity with such a fractional exponent as, when added to the fractional exponent of the given quantity, shall be equal to an integer.* Thus, $a^{\frac{3}{2}} \times a^{\frac{1}{2}} = a^{\frac{4}{2}} = a^2$. If the radical signs are to be used throughout, we must multiply by such a radical, of the same degree, as will make the quantity under the radical a perfect power of the same degree as the radical. Thus, $\sqrt[3]{a^4} \times \sqrt[3]{a^2} = \sqrt[3]{a^6} = a^2$.

(Art. 247, page 206.)

2. $(a + \sqrt{b})\,(a - \sqrt{b}) = a^2 - b$.

3. $(\sqrt{5} - \sqrt{1})\,(\sqrt{5} + \sqrt{1}) = 5 - 1 = 4$.

Note. It will be observed that the radical sign over 1 does not change its value, and the multiplication may be thus expressed:

$$(\sqrt{5} - 1)\,(\sqrt{5} + 1) = 5 - 1 = 4.$$

(ART. 249, page 207.)

2. $\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{a}{\sqrt{ab}}$.

3. $\frac{1}{\sqrt{3}+1} = \frac{1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{\sqrt{3}-1}{3-1} = \frac{\sqrt{3}-1}{2}$.

4. $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$, Ans.

5. $\frac{a}{\sqrt{b}+\sqrt{c}} = \frac{a}{\sqrt{b}+\sqrt{c}} \times \frac{\sqrt{b}-\sqrt{c}}{\sqrt{b}-\sqrt{c}} = \frac{a(\sqrt{b}-\sqrt{c})}{b-c}$.

6. $\frac{\sqrt{2}}{3-\sqrt{2}} = \frac{\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{3\sqrt{2}+2}{9-2} = \frac{3\sqrt{2}+2}{7}$.

7. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{(\sqrt{x}+\sqrt{y})^2}{x-y}$.

RADICAL EQUATIONS

LEADING TO SIMPLE EQUATIONS.

(ART. 254, page 210.)

4. Given $\sqrt{x+1} - 2 = 3$
Transposing and uniting, $\sqrt{x+1} = 5$
Squaring, $x + 1 = 25$
Transposing and uniting, $x = 24$

5. Given $\sqrt{x+7} = \sqrt{x} + 1$
Squaring, $x + 7 = x + 2\sqrt{x} + 1$
Transposing and uniting, $-2\sqrt{x} = -6$
Dividing by -2, $\sqrt{x} = 3$
Squaring, $x = 9$

6. Given $2 + \sqrt{3x - 30} = 5$
Transposing and uniting, $\sqrt{3x-30} = 3$
Squaring, $3x - 30 = 9$
Transposing and uniting, $3x = 39$
Whence, $x = 13$

7. Given $\sqrt{3-x}+6=8-1$
Transposing and uniting, $\sqrt{3-x}=1$
Squaring, $3-x=1$
Whence, $x=2$, Ans.

8. Given $x^{\frac{1}{2}}-7=-3$
Transposing and uniting, $x^{\frac{1}{2}}=4$
Squaring, $x=16$

9. Given $\sqrt{x+6}=3$
Squaring, $x+6=9$
Whence, $x=3$

10. Given $\sqrt{4+x}=4-\sqrt{x}$
Squaring, $4+x=16-8\sqrt{x}+x$
Transposing and uniting, $8\sqrt{x}=12$
Dividing by 8, $\sqrt{x}=\frac{3}{2}$
Squaring, $x=\frac{9}{4}=2\frac{1}{4}$

11. Given $y^{\frac{1}{3}}+5=11$
Transposing and uniting, $y^{\frac{1}{3}}=6$
Cubing, $y=216$

12. Given $\sqrt[4]{20-\sqrt{2x}}-2=0$
Transposing, $\sqrt[4]{20-\sqrt{2x}}=2$
Involving to the fourth power, $20-\sqrt{2x}=16$
Transposing and uniting, $-\sqrt{2x}=-4$
Squaring, $2x=16$
Whence, $x=8$

13. Given $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$
Clearing of fractions, $x^2-ax^2=x$
Dividing by x, $x-ax=1$
Whence, $x=\dfrac{1}{1-a}$

NOTE. Multiplying the given equation by $\sqrt{x}$ will also produce $x-ax=1$.

14. Given $x(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) = a x^{\frac{1}{2}}$

Multiplying by $x^{\frac{1}{2}}$, $x(x + 1) = a x$

Dividing by x, $x + 1 = a$

Transposing, $x = a - 1$

NOTE. Performing the multiplication indicated in the original equation gives $x^{\frac{3}{2}} + x^{\frac{1}{2}} = a x^{\frac{1}{2}}$, and this divided by $x^{\frac{1}{2}}$ also gives $x + 1 = a$.

QUADRATIC EQUATIONS.

PURE QUADRATIC EQUATIONS.

(ART. **263**, pp. 213, 214.)

3. Given $3x^2 - 2 = 2x^2 + 2$

Transposing and uniting, $x^2 = 4$

Evolving, $x = \pm 2$

4. Given $\frac{5x^2}{2} + x^2 = 126$

Clearing of fractions, $5x^2 + 2x^2 = 252$

Uniting terms, $7x^2 = 252$

Dividing by 7, $x^2 = 36$

Evolving, $x = \pm 6$

5. Given $y^2 = 9a^6$

Evolving, $y = \pm 3a^3$, Ans.

6. Given $7(2x^2 - 6) + 5(3 - x^2) = 198$

Expanding, $14x^2 - 42 + 15 - 5x^2 = 198$

Transposing and uniting, $9x^2 = 225$

Dividing by 9, $x^2 = 25$

Evolving, $x = \pm 5$

7. Given $(x + 1)^2 = 2x + 17$

Expanding, $x^2 + 2x + 1 = 2x + 17$

Transp. and uniting, $x^2 = 16$

Evolving, $x = \pm 4$

8. Given $x^2 + ab = 5x^2$

Transp. and uniting, $-4x^2 = -ab$

Dividing by -4, $x^2 = \frac{ab}{4}$

Evolving, $x = \pm \frac{\sqrt{ab}}{2} = \pm \frac{1}{2}\sqrt{ab}$

9 Given $\frac{2x^2 + 10}{15} = 7 - \frac{50 + x^2}{25}$

Multiplying by 75, $10x^2 + 50 = 525 - 150 - 3x^2$

Transp. and uniting, $13x^2 = 325$

Dividing by 13, $x^2 = 25$

Evolving, $x = \pm 5$

10. Given $(x + 2)^2 = 4x + 5$

Expanding, $x^2 + 4x + 4 = 4x + 5$

Transp. and uniting, $x^2 = 1$

Evolving, $x = \pm 1$

11. Given $(2x - 5)^2 = x^2 - 20x + 73$

Expanding, $4x^2 - 20x + 25 = x^2 - 20x + 73$

Transp. and uniting, $3x^2 = 48$

Dividing by 3, $x^2 = 16$

Evolving, $x = \pm 4$, Ans.

12. Given $4x - 150x^{-1} = x - 3x^{-1}$

Multiplying by x, $4x^2 - 150 = x^2 - 3$

Transp. and uniting, $3x^2 = 147$

Dividing by 3, $x^2 = 49$

Evolving, $x = \pm 7$

13 Given $\frac{3}{1 + x} + \frac{3}{1 - x} = 8$

Clear. of fractions, $3 - 3x + 3 + 3x = 8 - 8x^2$

Transp. and uniting, $8x^2 = 2$

Dividing by 8, $x^2 = \frac{1}{4}$

Evolving, $x = \pm \frac{1}{2}$

14. Given $x^2 - \frac{a^2}{4} + 3ab = 2a^2 + b^2$

Clear. of frac., $4x^2 - a^2 + 12ab = 8a^2 + 4b^2$

Transp. and uniting, $4x^2 = 9a^2 - 12ab + 4b^2$

Evolving, $2x = \pm (3a - 2b)$

Whence, $x = \pm \frac{3a - 2b}{2}$

Or, $x = \pm \left(\frac{3}{2}a - b\right)$

15. Given $c(x^2 + 4ab + 4bc) = a(a + 2c)^2 + dx^2 - a^2b$

Expanding first member,

$$cx^2 + 4abc + 4bc^2 = a(a + 2c)^2 + dx^2 - a^2b$$

Transposing terms,

$$cx^2 - dx^2 = a(a + 2c)^2 - a^2b - 4abc - 4bc^2$$

Factoring, $(c - d)x^2 = a(a + 2c)^2 - b(a + 2c)^2$

Or, $(c - d)x^2 = (a + 2c)^2(a - b)$

Dividing by $(c - d)$, $x^2 = \frac{(a + 2c)^2(a - b)}{c - d}$

Evolving, $x = \pm (a + 2c)\sqrt{\frac{a - b}{c - d}}$

16. Given $\frac{x + 2}{x - 2} + \frac{x - 2}{x + 2} = \frac{13}{6}$

Clear. of frac., $6(x + 2)^2 + 6(x - 2)^2 = 13(x^2 - 4)$

Expanding,

$$6x^2 + 24x + 24 + 6x^2 - 24x + 24 = 13x^2 - 52$$

Uniting terms, $12x^2 + 48 = 13x^2 - 52$

Transp. and uniting, $-x^2 = -100$

Changing signs, $x^2 = 100$

Evolving, $x = \pm 10$

17 Given $x - \frac{3x^2 - 2}{5x} = 3x^{-1} - \frac{2x^2 - 5}{3x}$

Mult. by $15x$, $15x^2 - 9x^2 + 6 = 45 - 10x^2 + 25$

Uniting terms, $6x^2 + 6 = 70 - 10x^2$

Transp. and uniting, $16x^2 = 64$

Dividing by 16, $x^2 = 4$

Evolving, $x = \pm 2$

18. Given $y^{-2} - \frac{2}{3y^2+1} = \frac{10}{9(3y^2+1)}$

Clearing of fractions, $9(3+y^{-2}) - 18 = 10$

Expanding, $27 + 9y^{-2} - 18 = 10$

Transp. and uniting, $9y^{-2} = 1$

Dividing by 9, $y^{-2} = \frac{1}{9}$

Evolving, $y^{-1} = \pm\frac{1}{3}$

Whence, $y = \pm 3$

19. Given $\frac{14z^2+16}{21} - \frac{2z^2+8}{8z^2-11} = \frac{2z^2}{3}$

Clearing of fractions,

$112z^4 - 26z^2 - 176 - 42z^2 - 168 = 112z^4 - 154z^2$

Transp. and uniting, $86z^2 = 344$

Dividing by 86, $z^2 = 4$

Evolving, $z = \pm 2$

(Art. 264, page 215.)

1. Given $22\frac{1}{2} + \sqrt{5(4x^2-1)} = 25$

Transp. and uniting, $\sqrt{5(4x^2-1)} = \frac{5}{2}$

Squaring, $5(4x^2-1) = \frac{25}{4}$

Dividing by 5, $4x^2 - 1 = \frac{5}{4}$

Transposing and uniting, $4x^2 = \frac{9}{4}$

Dividing by 4, $x^4 = \frac{9}{16}$

Evolving, $x = \pm\frac{3}{4}$

2. Given $\sqrt{x^2 + \sqrt{x^4 - a^4}} = a$

Squaring, $x^2 + \sqrt{x^4 - a^4} = a^2$

Transposing, $\sqrt{x^4 - a^4} = a^2 - x^2$

Squaring, $x^4 - a^4 = a^4 - 2a^2x^2 + x^4$

Transp. and uniting, $2a^2x^2 = 2a^4$

Dividing by $2a^2$, $x^2 = a^2$

Evolving, $x = \pm a$

3. Given $\sqrt{x^2 - 16} = \frac{3x}{5}$

Squaring, $x^2 - 16 = \frac{9x^2}{25}$

Clear. of fract., $25x^2 - 16 \times 25 = 9x^2$
Transposing and uniting, $16x^2 = 16 \times 25$
Dividing by 16, $x^2 = 25$
Evolving, $x = \pm 5$, Ans.

4. Given $\sqrt{a^2 + x^2} = \sqrt[4]{b^4 + x^4}$
Inv. to the fourth power, $a^4 + 2a^2x^2 + x^4 = b^4 + x^4$
Transposing and canceling, $2a^2x^2 = b^4 - a^4$
Dividing by $2a^2$, $x^2 = \frac{b^4 - a^4}{2a^2}$
Evolving, $x = \pm\sqrt{\frac{b^4 - a^4}{2a^2}}$

5. Given $\sqrt{\frac{4x^2 + 128}{3x}} = 2\sqrt{x}$
Squaring, $\frac{4x^2 + 128}{3x} = 4x$
Clearing of fractions, $4x^2 + 128 = 12x^2$
Whence, $8x^2 = 128$
Dividing by 8, $x^2 = 16$
Evolving, $x = \pm 4$

6. Given $\sqrt{x + a} = \sqrt{x + \sqrt{b^2 + x^2}}$
Squaring, $x + a = x + \sqrt{b^2 + x^2}$
Canceling x, $a = \sqrt{b^2 + x^2}$
Squaring, $a^2 = b^2 + x^2$
Whence, $x^2 = a^2 - b^2$
Evolving, $x = \pm\sqrt{a^2 - b^2}$

7. Given $x(10 + x^2)^{\frac{1}{2}} = 5 - x^2$
Squaring, $10x^2 + x^4 = 25 - 10x^2 + x^4$
Transposing and uniting, $20x^2 = 25$
Dividing by 20, $x^2 = \frac{5}{4}$
Evolving, $x = \pm\sqrt{\frac{5}{4}} = \pm\frac{1}{2}\sqrt{5}$

8. Given $x + (a^2 + x^2)^{\frac{1}{2}} = 2a^2(a^2 + x^2)^{-\frac{1}{2}}$
Mult. by $(a^2 + x^2)^{\frac{1}{2}}$, $x(a^2 + x^2)^{\frac{1}{2}} + a^2 + x^2 = 2a^2$

Transp. and uniting, $x(a^2+x^2)^{\frac{1}{2}}=a^2-x^2$

Squaring, $a^2x^2+x^4=a^4-2a^2x^2+x^4$

Transposing and uniting, $3a^2x^2=a^4$

Dividing by $3a^2$, $x^2=\frac{a^2}{3}$

Evolving, $x=\pm\frac{a}{\sqrt{3}}$

9. Given $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$

Rationaliz. denom., $\frac{(a-\sqrt{a^2-x^2})^2}{a^2-(a^2-x^2)}=b$

Or, $\frac{(a-\sqrt{a^2-x^2})^2}{x^2}=b$

Evolving, $\frac{a-\sqrt{a^2-x^2}}{x}=\pm\sqrt{b}$

Clear. of fractions, $a-\sqrt{a^2-x^2}=\pm x\sqrt{b}$

Transposing, $-\sqrt{a^2-x^2}=-a\pm x\sqrt{b}$

Squaring, $a^2-x^2=a^2\mp 2ax\sqrt{b}+bx^2$

Transp. and chang. signs, $x^2+bx^2=\pm 2ax\sqrt{b}$

Dividing by x, $x+bx=\pm 2a\sqrt{b}$

Dividing by $1+b$, $x=\pm\frac{2a\sqrt{b}}{1+b}$

10. Given $\sqrt{\frac{x+3}{x-3}}+\sqrt{\frac{x-3}{x+3}}=5$

Squaring, $\frac{x+3}{x-3}+2+\frac{x-3}{x+3}=25$

Transp. and uniting, $\frac{x+3}{x-3}+\frac{x-3}{x+3}=23$

Clearing of fractions, $(x+3)^2+(x-3)^2=23(x^2-9)$

Or, $x^2+6x+9+x^2-6x+9=23x^2-207$

Uniting terms, $2x^2+18=23x^2-207$

Whence, $21x^2=225$

Dividing by 21, $x^2=\frac{225}{21}=\frac{75}{7}$

Evolving, $x=\pm\sqrt{\tfrac{75}{7}}=\pm 5\sqrt{\tfrac{3}{7}}$

Or, $x=\pm 5\sqrt{\tfrac{21}{49}}=\pm\tfrac{5}{7}\sqrt{21}$

SIMULTANEOUS EQUATIONS.

(ART. 265, page 216.)

2. Given $\begin{cases} \frac{1}{3}x^2 - 3y^2 = 21 & (1) \\ \frac{1}{2}x + 2y = 0 & (2) \end{cases}$

Clearing (1) of fractions, $x^2 - 9y^2 = 63$ (3)

Clearing (2) of fractions, $x + 4y = 0$ (4)

From (4), $x = -4y$ (5)

Squaring (5), $x^2 = 16y^2$ (6)

Substituting (6) in (3), $16y^2 - 9y^2 = 63$ (7)

Or, $7y^2 = 63$ (8)

Dividing by 7, $y^2 = 9$ (9)

Evolving, $y = \pm 3$ (10)

Substituting in (5), $x = (-4) \times (\pm 3) = \mp 12$ (11)

NOTE. The values of x and y must have different signs; but it matters not which one is $\pm$, and which is $\mp$. (See NOTE, Art. 286, Alg.)

3. Given $\begin{cases} 5xy - 3y^2 = 100 & (1) \\ 5x - 4y = 0 & (2) \end{cases}$

Multiplying (2) by y, $5xy - 4y^2 = 0$ (3)

Subtracting (3) from (1), $y^2 = 100$ (4)

Evolving, $y = \pm 10$ (5)

Substituting in (2), $5x \mp 40 = 0$ (6)

Transposing, $5x = \pm 40$ (7)

Whence, $x = \pm 8$ (8)

4. Given $\begin{cases} xy + y^2 = 126 & (1) \\ 5y = 2x & (2) \end{cases}$

From (2), $x = \dfrac{5y}{2}$ (3)

Substituting (3) in (1), $\dfrac{5y^2}{2} + y^2 = 126$ (4)

Clearing of fractions, $5y^2 + 2y^2 = 252$ (5)

Uniting terms, $7y^2 = 252$ (6)

Dividing by 7, $y^2 = 36$ (7)

$$\begin{array}{lrl}
\text{Evolving,} & \left.\begin{array}{r} y = \pm 6 \\ x = \dfrac{\pm 30}{2} = \pm 15 \end{array}\right\} \text{Ans.} & \begin{array}{r}(8)\\(9)\end{array} \\
\text{Substituting in (3),} & &
\end{array}$$

5. Given $\left\{\begin{array}{r} 4x^2 + 7y^2 = 148 \\ 3x^2 - y^2 = 11 \end{array}\right.$ (1) (2)

Multiplying (2) by 7,	$21x^2 - 7y^2 = 77$	(3)
Adding (1) and (3),	$25x^2 = 225$	(4)
Dividing by 25,	$x^2 = 9$	(5)
Evolving,	$x = \pm 3$	(6)
Substituting (5) in (2),	$27 - y^2 = 11$	(7)
Whence,	$y^2 = 16$	(8)
Evolving,	$y = \pm 4$	(9)

6. Given $\left\{\begin{array}{l} x + y = 3x - 3y \\ x^3 - y^3 = 56 \end{array}\right.$ (1) (2)

Transposing in (1),	$-2x = -4y$	(3)
Dividing by -2,	$x = 2y$	(4)
Cubing (4),	$x^3 = 8y^3$	(5)
Substituting (5) in (2),	$8y^3 - y^3 = 56$	(6)
Uniting terms,	$7y^3 = 56$	(7)
Dividing by 7,	$y^3 = 8$	(8)
Evolving,	$y = 2$	(9)
Substituting in (4),	$x = 2 \times 2 = 4$	(10)

7. Given $\left\{\begin{array}{r} x^2 + y^2 : x^2 - y^2 :: 17 : 8 \\ xy^2 = 45 \end{array}\right.$ (1) (2)

From (1),	$8x^2 + 8y^2 = 17x^2 - 17y^2$	(3)
From (3),	$9x^2 = 25y^2$	(4)
Evolving,	$3x = \pm 5y$	(5)
From (5),	$x = \pm \dfrac{5y}{3}$	(6)
Substituting (6) in (2),	$\pm \dfrac{5y^3}{3} = 45$	(7)
Clearing of fractions,	$\pm 5y^3 = 135$	(8)
Dividing by ± 5,	$y^3 = \pm 27$	(9)
Evolving,	$y = \pm 3$	(10)
Substituting in (6),	$x = \pm \dfrac{5 \times (\pm 3)}{3} = 5$	(11)

NOTE. The value of x can only take the positive sign, for $y^2 = 9$, which, substituted in (2), gives $9x = 45$, and $x = 5$. The same results are obtained if we first eliminate y^2 from (2) and (4).

PROBLEMS

LEADING TO PURE EQUATIONS.

(ART. 266, page 217.)

2. Let $x =$ the smaller number,
and $5x =$ the larger number.

Then, $25x^2 - x^2 = 96$
Uniting terms, $24x^2 = 96$
Dividing by 24, $x^2 = 4$
Evolving, $x = 2$, the smaller number.
Then, $5x = 10$, the larger number.

3. Let $3x =$ length of the field,
and $2x =$ breadth of the field.

Then, $6x^2 = 3$ A. 3 R. $= 600$ rods.
Dividing by 6, $x^2 = 100$
Evolving, $x = 10$
Then, $3x = 30$, the length.
Also, $2x = 20$, the breadth.

4. Let $x =$ no. yards in first piece,
and $y =$ no. yards in second piece.
Then $x^2 =$ entire price of first, in dimes,
and $y^2 =$ entire price of second, in dimes

Therefore, $x + y = 36$ (1)
and, $x^2 : y^2 :: 4 : 1$ (2)
From (2), $x^2 = 4y^2$ (3)
Evolving, $x = 2y$ (4)
Substituting (4) in (1), $2y + y = 36$ (5)
Uniting terms, $3y = 36$ (6)
Dividing by 3, $y = 12$ (7)
Substituting in (4), $x = 2 \times 12 = 24$ (8)

Note. If only one unknown quantity be used, the equation may be derived from the proportion $x^2 : (36 - x)^2 :: 4 : 1$.

AFFECTED QUADRATIC EQUATIONS.

FIRST METHOD OF COMPLETING THE SQUARE.

(Art. **269**, pp. 223, 224.)

6. Given $x^2 + 2x = 8$
Compl. square by adding $(\frac{2}{2})^2$, $x^2 + 2x + 1 = 9$
Evolving, $x + 1 = \pm 3$
Transposing, $x = -1 \pm 3$
Uniting, $x = 2$, or -4

7. Given $x^2 - 4x = -4$
Completing the square, $x^2 - 4x + 4 = 0$
Evolving, $x - 2 = \pm 0$
Transposing, $x = 2 \pm 0$
Or, $x = 2$, or 2

8. Given $x^2 - 6x = 55$
Compl. the square, $x^2 - 6x + 9 = 64$
Evolving, $x - 3 = \pm 8$
Transposing, $x = 3 \pm 8$
Uniting, $x = 11$, or -5, Ans.

9. Given $x^2 + 12x + 35 = 0$
Transposing, $x^2 + 12x = -35$
Completing the square, $x^2 + 12x + 36 = 1$
Evolving, $x + 6 = \pm 1$
Whence, $x = -6 \pm 1 = -5$, or -7

10. Given $3z^2 + 48 = 30z$
Transposing terms, $3z^2 - 30z = -48$

Dividing by 3, $z^2 - 10z = -16$
Completing the square, $z^2 - 10z + 25 = 9$
Evolving, $z - 5 = \pm 3$
Whence, $z = 5 \pm 3 = 8$, or 2

11. Given $x^2 - 2ax = b$
Compl. the square, $x^2 - 2ax + a^2 = a^2 + b$
Evolving, $x - a = \pm \sqrt{a^2 + b}$
Transposing, $x = a \pm \sqrt{a^2 + b}$

12. Given $x^2 = 3x + 10$
Or, $x^2 - 3x = 10$
Compl. the square, $x^2 - 3x + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4}$
Evolving, $x - \frac{3}{2} = \pm \frac{7}{2}$
Whence, $x = \frac{3}{2} \pm \frac{7}{2} = 5$, or -2

13. Given $2x + 60 = 2x^2$
Or, $2x^2 - 2x = 60$
Dividing by 2, $x^2 - x = 30$
Compl. the square, $x^2 - x + \frac{1}{4} = 30\frac{1}{4} = \frac{121}{4}$
Evolving, $x - \frac{1}{2} = \pm \frac{11}{2}$
Whence, $x = \frac{1}{2} \pm \frac{11}{2} = 6$, or -5

14. Given $4y^2 + 8y = 5$
Dividing by 4, $y^2 + 2y = \frac{5}{4}$
Compl. the square, $y^2 + 2y + 1 = \frac{5}{4} + 1 = \frac{9}{4}$
Evolving, $y + 1 = \pm \frac{3}{2}$
Whence, $y = -1 \pm \frac{3}{2} = \frac{1}{2}$ or $-\frac{5}{2}$

15. Given $5x^2 + 20 = 25x$
Transp. and div. by 5, $x^2 - 5x = -4$
Compl. the square, $x^2 - 5x + \frac{25}{4} = \frac{25}{4} - 4 = \frac{9}{4}$
Evolving, $x - \frac{5}{2} = \pm \frac{3}{2}$
Whence, $x = \frac{5}{2} \pm \frac{3}{2} = 4$, or 1

16. Given $3x + 4 = 39x^{-1}$
Multiplying by x, $3x^2 + 4x = 39$
Dividing by 3, $x^2 + \frac{4}{3}x = 13$

Compl. the square, $x^2 + \frac{4}{3}x + \frac{4}{9} = 13\frac{4}{9} = \frac{121}{9}$
Evolving, $x + \frac{2}{3} = \pm \frac{11}{3}$
Whence, $x = -\frac{2}{3} \pm \frac{11}{3} = 3$, or $-\frac{13}{3}$

17. Given $5x^2 - 40x = 70$
Dividing by 5, $x^2 - 8x = 14$
Completing the square, $x^2 - 8x + 16 = 30$
Evolving, $x - 4 = \pm \sqrt{30}$
Transposing, $x = 4 \pm \sqrt{30}$

18. Given $3x = 10 + \frac{1}{4}x^2$
Transposing, $-\frac{1}{4}x^2 + 3x = 10$
Mult. by -4, $x^2 - 12x = -40$
Compl. the square, $x^2 - 12x + 36 = -4$
Evolving, $x - 6 = \pm\sqrt{-4} = \pm 2\sqrt{-1}$
Transposing, $x = 6 \pm 2\sqrt{-1}$

Note. Multiplying by -4, is the same as dividing by $-\frac{1}{4}$, the coefficient of x^2, according to the Rule and Note 1.

19. Given $x^2 - 6x = 0$
Compl. the square, $x^2 - 6x + 9 = 9$
Evolving, $x - 3 = \pm 3$
Whence, $x = 3 \pm 3 = 6$, or 0

Note. By dividing the original equation by x, and transposing, we obtain at once $x = 6$.

20. Given $a^{-1}x + ax^{-1} = 2a^{-1}$
Multiplying by ax, $x^2 + a^2 = 2x$
Transposing terms, $x^2 - 2x = -a^2$
Compl. the square, $x^2 - 2x + 1 = 1 - a^2$
Evolving, $x - 1 = \pm\sqrt{1 - a^2}$
Transposing, $x = 1 \pm \sqrt{1 - a^2}$

(Art. 270, page 225.)

1. In this equation, $p = -8$, and $q = 9$; hence
$x = 4 \pm \sqrt{9 + (-4)^2} = 4 \pm \sqrt{25} = 4 \pm 5 = 9$, or -1.

2. In this equation, $p = 16$, and $q = -55$; hence
$x = -8 \pm \sqrt{-55 + 8^2} = -8 \pm \sqrt{9} = -8 \pm 3$
$= -5$, or -11.

3. In this equation, $p = -20$, and $q = 800$; hence
$x = 10 \pm \sqrt{800 + (-10)^2} = 10 \pm \sqrt{900} = 10 \pm 30$
$= 40$, or -20. Ans. $x = 40$, or -20.

4. In this equation, $p = 5$, and $q = 14$; hence
$x = -\frac{5}{2} \pm \sqrt{14 + (\frac{5}{2})^2} = -\frac{5}{2} \pm \sqrt{\frac{81}{4}} = -\frac{5}{2} \pm \frac{9}{2}$
$= 2$, or -7.

5. Multiplying the given equation by 3, which is the same as dividing by $\frac{1}{3}$, the coefficient of x^2, we have
$x^2 + \frac{9}{2}x = 63$, in which $p = \frac{9}{2}$, and $q = 63$; hence
$x = -\frac{9}{4} \pm \sqrt{63 + (\frac{9}{4})^2} = -\frac{9}{4} \pm \sqrt{\frac{1089}{16}} = -\frac{9}{4} \pm \frac{33}{4}$
$= 6$, or $-10\frac{1}{2}$.

6. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$
Multiplying by 2, $x^2 - \frac{2}{3}x + 14\frac{3}{4} = 16$
Transp. and uniting, $x^2 - \frac{2}{3}x = 1\frac{1}{4} = \frac{5}{4}$
In this equation, $p = -\frac{2}{3}$, and $q = \frac{5}{4}$; hence
$x = \frac{1}{3} \pm \sqrt{\frac{5}{4} + \frac{1}{9}} = \frac{1}{3} \pm \sqrt{\frac{49}{36}} = \frac{2}{6} \pm \frac{7}{6} = \frac{3}{2}$, or $-\frac{5}{6}$.

SECOND METHOD OF COMPLETING THE SQUARE.

(Art. 272, page 229.)

4 Given $x^2 - 7x + 6 = 0$
Transposing, $x^2 - 7x = -6$
Multiplying by 4, $4x^2 - 28x = -24$
Completing the square,
$$4x^2 - 28x + 49 = 25$$
Evolving, $2x - 7 = \pm 5$
Transposing, $2x = 7 \pm 5 = 12$, or 2
Dividing by 2, $x = 6$, or 1

5. Given $x^2 + \frac{x}{2} = 3$

Clearing of fractions, $2x^2 + x = 6$

Multiplying by 8, $16x^2 + 8x = 48$

Completing the square, $16x^2 + 8x + 1 = 49$

Evolving, $4x + 1 = \pm 7$

Whence, $4x = -1 \pm 7 = 6$, or -8

Dividing by 4, $x = 1\frac{1}{2}$, or -2

6. Given $10x = 6x^2 + 4$

Transposing, $-6x^2 + 10x = 4$

Dividing by -2, $3x^2 - 5x = -2$

Multiplying by 12, $36x^2 - 60x = -24$

Compl. square, $36x^2 - 60x + 25 = 1$

Evolving, $6x - 5 = \pm 1$

Whence, $6x = 5 \pm 1 = 6$, or 4

Dividing by 6, $x = 1$, or $\frac{2}{3}$, Ans.

7. Given $5x^2 = 57 - 4x$

Transposing, $5x^2 + 4x = 57$

Multiplying by 5, $25x^2 + 20x = 285$

Compl. square, $25x^2 + 20x + 4 = 289$

Evolving, $5x + 2 = \pm 17$

Whence, $5x = -2 \pm 17 = 15$, or -19

Dividing by 5, $x = 3$, or $-3\frac{4}{5}$

NOTE. As 4, the coefficient of x, is an *even* number, we complete the square by Note 2.

8. Given $\frac{a^2 - b^2}{c} = 2ax - cx^2$

Transposing, $cx^2 - 2ax = \frac{b^2 - a^2}{c}$

Multiplying by c, $c^2x^2 - 2acx = b^2 - a^2$

Compl. the square, $c^2x^2 - 2acx + a^2 = b^2$

Evolving, $cx - a = \pm b$

Transposing, $cx = a \pm b$

Dividing by c, $x = \frac{a \pm b}{c}$

Note. By multiplying the equation by c, we both clear of fractions and make the first term a perfect square. As $2a$ is *even*, we use c instead of $4c$, according to Note 2.

9. Given $\frac{x}{a} + b x^{-1} = c$

Multiplying by $a x$, $x^2 + ab = acx$

Transposing, $x^2 - acx = -ab$

Multiplying by 4, $4x^2 - 4acx = -4ab$

Compl. the square, $4x^2 - 4acx + a^2c^2 = a^2c^2 - 4ab$

Evolving, $2x - ac = \pm\sqrt{a^2c^2 - 4ab}$

Transposing, $2x = ac \pm \sqrt{a^2c^2 - 4ab}$

Dividing by 2, $x = \frac{ac \pm \sqrt{a^2c^2 - 4ab}}{2}$

(Art. 273, pp. 230, 231.)

1. Given $x^2 - 14x = 120$

Adding $(\frac{14}{2})^2$, $x^2 - 14x + 49 = 169$

Evolving, $x - 7 = \pm 13$

Transposing, $x = 7 \pm 13$

Uniting, $x = 20$, or -6

2. Given $x^2 - \frac{3x}{2} = 27$

Clearing of fractions, $2x^2 - 3x = 54$

Multiplying by 8, $16x^2 - 24x = 432$

Adding 3^2, $16x^2 - 24x + 9 = 441$

Evolving, $4x - 3 = \pm 21$

Whence, $4x = 3 \pm 21 = 24$, or -18

Dividing by 4, $x = 6$, or $-4\frac{1}{2}$

3. Given $2x^2 - 10x = 100$

Dividing by 2, $x^2 - 5x = 50$

Adding $(\frac{5}{2})^2$, $x^2 - 5x + \frac{25}{4} = 50 + \frac{25}{4} = \frac{225}{4}$

Evolving, $x - \frac{5}{2} = \pm\frac{15}{2}$

Transposing, $x = \frac{5}{2} \pm \frac{15}{2}$

Uniting, $x = 10$, or -5, Ans.

4. Given $16x^{-1} - 4 = 12x^{-2}$

Dividing by 4, $4x^{-1} - 1 = 3x^{-2}$

Transposing, &c., $3x^{-2} - 4x^{-1} = -1$

Multiplying by 3, $9x^{-2} - 12x^{-1} = -3$

Adding $\left(\frac{4}{2}\right)^2$, $9x^{-2} - 12x^{-1} + 4 = 1$

Evolving, $3x^{-1} - 2 = \pm 1$

Whence, $3x^{-1} = 2 \pm 1 = 3$, or 1

Dividing by 3, $x^{-1} = 1$, or $\frac{1}{3}$

Hence, $x = 1$, or 3

5. Given $x^2 - \frac{3}{10}x = \frac{1}{10}$

Adding $\left(\frac{3}{20}\right)^2$, $x^2 - \frac{3}{10}x + \frac{9}{400} = \frac{1}{10} + \frac{9}{400} = \frac{49}{400}$

Evolving, $x - \frac{3}{20} = \pm \frac{7}{20}$

Transposing, $x = \frac{3}{20} \pm \frac{7}{20}$

Uniting, $x = \frac{1}{2}$, or $-\frac{1}{5}$

6. Given $\frac{2x^2}{3} + 3\frac{1}{2} = \frac{x}{2} + 8$

Multiplying by 6, $4x^2 + 21 = 3x + 48$

Transp. and uniting, $4x^2 - 3x = 27$

Multiplying by 16, $64x^2 - 48x = 432$

Adding 3^2, $64x^2 - 48x + 9 = 441$

Evolving, $8x - 3 = \pm 21$

Whence, $8x = 3 \pm 21 = 24$, or -18

Dividing by 8, $x = 3$, or $-\frac{9}{4}$, Ans.

7. Given $\frac{z^2}{12} + \frac{20}{3} = 2z$

Clearing of Fractions, $z^2 + 80 = 24z$

Transposing, $z^2 - 24z = -80$

Adding $\left(\frac{24}{2}\right)^2$ $z^2 - 24z + 144 = 64$

Evolving, $z - 12 = \pm 8$

Whence, $z = 12 \pm 8$

Or, $z = 20$, or 4

8. Given $2x^2 + 15 = 3x$

Transposing, $2x^2 - 3x = -15$

Multiplying by 8, $16x^2 - 24x = -120$

Adding 3^2, $16x^2 - 24x + 9 = -120 + 9 = -111$

Evolving, $4x - 3 = \pm\sqrt{-111}$

Transposing, $4x = 3 \pm \sqrt{-111}$

Dividing by 4, $x = \dfrac{3 \pm \sqrt{-111}}{4}$

NOTE. These values are *imaginary*.

9. Given $x^2 - 6x + 19 = 13$

Transposing and uniting, $x^2 - 6x = -6$

Adding $(\frac{6}{2})^2$, $x^2 - 6x + 9 = -6 + 9 = 3$

Evolving, $x - 3 = \pm\sqrt{3}$

Transposing, $x = 3 \pm \sqrt{3}$

Or, $x = 3 \pm 1.732$

Hence, $x = 4.732$, or 1.268

10. Given $4ax^2 - 2bx = c$

Mult. by $4a$, $16a^2x^2 - 8abx = 4ac$

Adding $\left(\frac{2b}{2}\right)^2$, $16a^2x^2 - 8abx + b^2 = 4ac + b^2$

Evolving, $4ax - b = \pm\sqrt{4ac + b^2}$

Transposing, $4ax = b \pm \sqrt{4ac + b^2}$

Dividing by $4a$, $x = \dfrac{b \pm \sqrt{4ac + b^2}}{4a}$

11. Given $x^2 - 4 = 16 - (x - 2)^2$

Expanding, $x^2 - 4 = 16 - x^2 + 4x - 4$

Transp. and uniting, $2x^2 - 4x = 16$

Dividing by 2, $x^2 - 2x = 8$

Adding $(\frac{2}{2})^2$, $x^2 - 2x + 1 = 9$

Evolving, $x - 1 = \pm 3$

Whence, $x = 1 \pm 3 = 4$, or -2

12. Given $(3x - 5)(2x - 5) = (x + 3)(x - 1)$

Expanding, $6x^2 - 25x + 25 = x^2 + 2x - 3$

Transp. and uniting, $5x^2 - 27x = -28$
Mult. by 20, $100x^2 - 540x = -560$
Adding 27^2, $100x^2 - 540x + 729 = 169$
Evolving, $10x - 27 = \pm 13$
Whence, $10x = 27 \pm 13 = 40, \text{ or } 14$
Dividing by 10, $x = 4, \text{ or } \frac{7}{5}$

13. Given $(2x - 3)^2 = 8x$
Expanding, $4x^2 - 12x + 9 = 8x$
Transp. and uniting, $4x^2 - 20x = -9$
Adding $\left(\frac{10}{2}\right)^2$ $4x^2 - 20x + 25 = 16$
Evolving, $2x - 5 = \pm 4$
Whence, $2x = 5 \pm 4 = 9, \text{ or } 1$
Dividing, $x = \frac{9}{2}, \text{ or } \frac{1}{2}$

14. Given $\frac{1}{6}(x - 3)^2 + \frac{9}{2} = x$
Expand. and mult. by 6, $x^2 - 6x + 9 + 27 = 6x$
Transposing and uniting, $x^2 - 12x = -36$
Adding $(\frac{12}{2})^2$, $x^2 - 12x + 36 = 0$
Evolving, $x - 6 = \pm 0$
Transposing, $x = 6, \text{ or } 6$

15. Given $x^2 + (x + 1)^2 = \frac{13}{6}x(x + 1)$
Expanding, $x^2 + x^2 + 2x + 1 = \frac{13x^2 + 13x}{6}$
Clear. of fract., $6x^2 + 6x^2 + 12x + 6 = 13x^2 + 13x$
Whence, $x^2 + x = 6$
Adding $(\frac{1}{2})^2$, $x^2 + x + \frac{1}{4} = 6\frac{1}{4} = \frac{25}{4}$
Evolving, $x + \frac{1}{2} = \pm \frac{5}{2}$
Whence, $x = -\frac{1}{2} \pm \frac{5}{2} = 2, \text{ or } -3$

16. Given $3(2 - x) + 2(3 - x) = 2(4 + 3x^2)$
Expanding, $6 - 3x + 6 - 2x = 8 + 6x^2$
Whence, $6x^2 + 5x = 4$
Multiplying by 24, $144x^2 + 120x = 96$

Adding 5^2, $144x^2 + 120x + 25 = 121$
Evolving, $12x + 5 = \pm 11$
Whence, $12x = -5 \pm 11 = 6$, or -16
Dividing by 12, $x = \frac{1}{2}$, or $-\frac{4}{3}$

17. Given $4(x-1) - \frac{x-1}{2x} = 3\frac{3}{4}$

Mult. by $4x$, $16x^2 - 16x - 2x + 2 = 15x$
Transp. and uniting, $16x^2 - 33x = -2$
Multiplying by 16, $256x^2 - 528x = -32$
Adding $\left(\frac{33}{2}\right)^2$ $256x^2 - 528x + \frac{1089}{4} = \frac{961}{4}$
Evolving, $16x - \frac{33}{2} = \pm \frac{31}{2}$
Whence, $16x = \frac{33}{2} \pm \frac{31}{2} = 32$, or 1
Dividing by 16, $x = 2$, or $\frac{1}{16}$

18. Given $\frac{6}{x+1} + \frac{2}{x} = 3$

Clear. of fractions, $6x + 2x + 2 = 3x^2 + 3x$
Whence, $3x^2 - 5x = 2$
Multiplying by 12, $36x^2 - 60x = 24$
Adding 5^2, $36x^2 - 60x + 25 = 49$
Evolving, $6x - 5 = \pm 7$
Whence, $6x = 5 \pm 7 = 12$, or -2
Dividing by 6, $x = 2$, or $-\frac{1}{3}$

19. Given $\frac{7}{x+1} + \frac{2}{x} - 5 = 0$

Clear. of fract., $7x + 2x + 2 - 5x^2 - 5x = 0$
Whence, $5x^2 - 4x = 2$
Multiplying by 5, $25x^2 - 20x = 10$
Adding $(\frac{4}{2})^2$, $25x^2 - 20x + 4 = 14$
Evolving, $5x - 2 = \pm \sqrt{14}$
Transposing, $5x = 2 \pm \sqrt{14}$
Dividing by 5, $x = \frac{2 \pm \sqrt{14}}{5}$
Or, $x = 1.148$, or -0.348

20. Given $\frac{x}{x+60} = \frac{7}{3x-5}$

Clearing of fractions, $3x^2 - 5x = 7x + 420$
Transp. and uniting, $3x^2 - 12x = 420$
Dividing by 3, $x^2 - 4x = 140$
Adding $(\frac{4}{2})^2$, $x^2 - 4x + 4 = 144$
Evolving, $x - 2 = \pm 12$
Whence, $x = 2 \pm 12 = 14$, or -10

21. Given $8x + 11 + 7x^{-1} = 3 + \frac{65x}{7}$

Multiplying by $7x$, $56x^2 + 77x + 49 = 21x + 65x^2$
Whence, $9x^2 - 56x = 49$
Multiplying by 36, $324x^2 - 2016x = 1764$
Adding 56^2, $324x^2 - 2016x + 3136 = 4900$
Evolving, $18x - 56 = \pm 70$
Whence, $18x = 56 \pm 70 = 126$ or -14
Dividing by 18, $x = 7$ or $-\frac{7}{9}$

22. Given $\frac{21}{5-x} - \frac{x}{7} = 3\frac{2}{7}$

Clearing of fractions, $147 - 5x + x^2 = 115 - 23x$
Transp. and uniting, $x^2 + 18x = -32$
Adding $(\frac{18}{2})^2$, $x^2 + 18x + 81 = 49$
Evolving, $x + 9 = \pm 7$
Whence, $x = -9 \pm 7 = -2$, or -16

23. Given $\frac{x^2 - 5x}{x+3} = x - 3 + x^{-1}$

Mult. by $x(x+3)$, $x^3 - 5x^2 = x^3 - 9x + x + 3$
Transp. and uniting, $-5x^2 + 8x = 3$
Mult. by -5, $25x^2 - 40x = -15$
Adding $(\frac{8}{2})^2$, $25x^2 - 40x + 16 = 1$
Evolving, $5x - 4 = \pm 1$
Whence, $5x = 4 \pm 1 = 5$, or 3
Dividing by 5, $x = 1$, or $\frac{3}{5}$

24. Given $$\frac{1}{x+6} + 8x^{-1} = \frac{3}{x+2}$$

Multiplying by $x(x+6)(x+2)$,

$$x^2 + 2x + 8x^2 + 64x + 96 = 3x^2 + 18x$$

Transp. and uniting, $6x^2 + 48x = -96$

Dividing by 6, $x^2 + 8x = -16$

Adding $(\frac{8}{2})^2$, $x^2 + 8x + 16 = 0$

Evolving, $x + 4 = \pm 0$

Whence, $x = -4 \pm 0 = -4$, or -4

25. Given $$\frac{x+3}{x+2} + \frac{x-3}{x-2} = \frac{2x-3}{x-1}$$

Clearing of fractions,

$$x^3 - 7x + 6 + x^3 - 2x^2 - 5x + 6 = 2x^3 - 3x^2 - 8x + 12$$

Transp. and uniting, $x^2 - 4x = 0$

Adding $(\frac{4}{2})^2$, $x^2 - 4x + 4 = 4$

Evolving, $x - 2 = \pm 2$

Whence, $x = 2 \pm 2 = 4$, or 0

26. Given $$(a+b)x^2 - cx = \frac{ac}{a+b}$$

Mult. by $4(a+b)$, $4(a+b)^2 x^2 - 4c(a+b)x = 4ac$

Adding c^2, $4(a+b)^2 x^2 - 4c(a+b)x + c^2 = c^2 + 4ac$

Evolving, $2(a+b)x - c = \pm \sqrt{c^2 + 4ac}$

Transposing, $2(a+b)x = c \pm \sqrt{c^2 + 4ac}$

Dividing by $2(a+b)$, $$x = \frac{c \pm \sqrt{c^2 + 4ac}}{2(a+b)}$$

27. Given $$\sqrt{x^5} + \sqrt{x^3} = 6\sqrt{x}$$

Dividing by $\sqrt{x}$, $x^2 + x = 6$

Adding $(\frac{1}{2})^2$, $x^2 + x + \frac{1}{4} = 6\frac{1}{4} = \frac{25}{4}$

Evolving, $x + \frac{1}{2} = \pm \frac{5}{2}$

Whence, $x = -\frac{1}{2} \pm \frac{5}{2} = 2$, or -3

28. Given $$(4x+5)^{\frac{1}{2}}(7x+1)^{\frac{1}{2}} = 30$$

Squaring, $(4x+5)(7x+1) = 900$

Expanding, $28x^2 + 39x + 5 = 900$
Transp. and uniting, $28x^2 + 39x = 895$
Dividing by 28, $x^2 + \frac{39}{28}x = \frac{895}{28}$
Adding $(\frac{39}{56})^2$, $x^2 + \frac{39}{28}x + (\frac{39}{56})^2 = \frac{101761}{3136}$
Evolving, $x + \frac{39}{56} = \pm \frac{319}{56}$
Transposing, $x = -\frac{39}{56} \pm \frac{319}{56}$
Uniting, $x = 5$, or $-\frac{179}{28}$

EQUATIONS IN THE QUADRATIC FORM.

(Art. 276, page 233.)

3 Given $x^4 - 9x^2 + 20 = 0$
Transposing, $x^4 - 9x^2 = -20$
Adding $(\frac{9}{2})^2$, $x^4 - 9x^2 + \frac{81}{4} = \frac{1}{4}$
Extracting square root, $x^2 - \frac{9}{2} = \pm \frac{1}{2}$
Whence, $x^2 = \frac{9}{2} \pm \frac{1}{2} = 5$, or 4
Extracting square root, $x = \pm \sqrt{5}$, or ± 2

4. Given $x^6 - 35x^3 + 216 = 0$
Transposing, $x^6 - 35x^3 = -216$
Adding $(\frac{35}{2})^2$, $x^6 - 35x^3 + \frac{1225}{4} = \frac{361}{4}$
Extracting square root, $x^3 - \frac{35}{2} = \pm \frac{19}{2}$
Whence, $x^3 = \frac{35}{2} \pm \frac{19}{2} = 27$, or 8
Extracting cube root, $x = 3$, or 2

5. Given $5x^6 - 90x^3 - 270 = 945$
Transp. and unit., $5x^6 - 90x^3 = 1215$
Dividing by 5, $x^6 - 18x^3 = 243$
Adding $(\frac{18}{2})^2$, $x^6 - 18x^3 + 81 = 324$
Extracting square root, $x^3 - 9 = \pm 18$
Whence, $x^3 = 9 \pm 18 = 27$, or -9
Extracting cube root, $x = 3$, or $\sqrt[3]{-9}$

6. Given $x^{10} + 31x^5 = 32$
Multiplying by 4, $4x^{10} + 124x^5 = 128$
Adding $(31)^2$, $4x^{10} + 124x^5 + 961 = 1089$
Extracting square root, $2x^5 + 31 = \pm 33$
Whence, $2x^5 = -31 \pm 33 = 2$, or -64

Dividing by 2, $x^5 = 1$, or -32
Extracting fifth root, $x = 1$, or -2

7. Given $x^{2n} - 4x^n = 10$
Adding $(\frac{4}{2})^2$, $x^{2n} - 4x^n + 4 = 14$
Extracting square root, $x^n - 2 = \pm\sqrt{14}$
Transposing, $x^n = 2 \pm \sqrt{14}$
Extracting nth root, $x = (2 \pm \sqrt{14})^{\frac{1}{n}}$

8. Given $x^2 + 1225x^{-2} = 74$
Multiplying by x^2, $x^4 + 1225 = 74x^2$
Transposing, $x^4 - 74x^2 = -1225$
Adding $(\frac{74}{2})^2$, $x^4 - 74x^2 + 1369 = 144$
Extracting square root, $x^2 - 37 = \pm 12$
Whence, $x^2 = 37 \pm 12 = 49$, or 25
Extracting square root, $x = \pm 7$, or ± 5

(ART. 277, page 235.)

3. Given $x + 4\sqrt{x} = 21$
Adding $(\frac{4}{2})^2$, $x + 4\sqrt{x} + 4 = 25$
Extracting square root, $\sqrt{x} + 2 = \pm 5$
Whence, $\sqrt{x} = -2 \pm 5 = 3$, or -7
Squaring, $x = 9$, or 49

4. Given $x^{-1} + x^{-\frac{1}{2}} = 6$
Adding $(\frac{1}{2})^2$, $x^{-1} + x^{-\frac{1}{2}} + \frac{1}{4} = 6\frac{1}{4} = \frac{25}{4}$
Extracting square root, $x^{-\frac{1}{2}} + \frac{1}{2} = \pm \frac{5}{2}$
Whence, $x^{-\frac{1}{2}} = -\frac{1}{2} \pm \frac{5}{2} = 2$, or -3
Squaring, $x^{-1} = 4$, or 9
Hence, $x = \frac{1}{4}$, or $\frac{1}{9}$

5. Given $x^{\frac{4}{3}} + 10x^{\frac{2}{3}} = 171$
Adding $(\frac{10}{2})^2$, $x^{\frac{4}{3}} + 10x^{\frac{2}{3}} + 25 = 196$

Extracting square root, $x^{\frac{2}{3}} + 5 = \pm 14$

Whence, $x^{\frac{2}{3}} = -5 \pm 14 = 9$, or -19

By Note, $x = 9^{\frac{3}{2}}$, or $(-19)^{\frac{3}{2}}$

Or, $x = 27$, or $(-19)^{\frac{3}{2}}$

NOTE. It will be observed that the second value of x is *imaginary*. (Art. 250.)

6. Given $5y^{\frac{1}{2}} + y^{\frac{1}{4}} = 22$

Multiplying by 20, $100y^{\frac{1}{2}} + 20y^{\frac{1}{4}} = 440$

Adding 1, $100y^{\frac{1}{2}} + 20y^{\frac{1}{4}} + 1 = 441$

Extracting square root, $10y^{\frac{1}{4}} + 1 = \pm 21$

Whence, $10y^{\frac{1}{4}} = -1 \pm 21 = 20$, or -22

Dividing by 10, $y^{\frac{1}{4}} = 2$, or $-\frac{11}{5}$

Raising to fourth power, $y = 16$, or $(-\frac{11}{5})^4$

7. Given $\sqrt[5]{x} + \sqrt[5]{x^2} = 6$

Or, $x^{\frac{2}{5}} + x^{\frac{1}{5}} = 6$

Adding $(\frac{1}{2})^2$, $x^{\frac{2}{5}} + x^{\frac{1}{5}} + \frac{1}{4} = 6\frac{1}{4} = \frac{25}{4}$

Extract. square root, $x^{\frac{1}{5}} + \frac{1}{2} = \pm \frac{5}{2}$

Whence, $x^{\frac{1}{5}} = -\frac{1}{2} \pm \frac{5}{2} = 2$, or -3

Raising to fifth power, $x = 32$, or -243

8. Given $x^{\frac{1}{n}} - x^{\frac{2}{n}} + 2 = 0$

Or, $x^{\frac{2}{n}} - x^{\frac{1}{n}} = 2$

Adding $(\frac{1}{2})^2$, $x^{\frac{2}{n}} - x^{\frac{1}{n}} + \frac{1}{4} = 2\frac{1}{4} = \frac{9}{4}$

Extracting square root, $x^{\frac{1}{n}} - \frac{1}{2} = \pm \frac{3}{2}$

Whence, $x^{\frac{1}{n}} = \frac{1}{2} \pm \frac{3}{2} = 2$, or -1

Raising to nth power, $x = 2^n$, or $(-1)^n$

9. Given $x^n + p x^{\frac{n}{2}} = q$

Adding $\left(\frac{p}{2}\right)^2$, $x^n + p x^{\frac{n}{2}} + \frac{1}{4} p^2 = q + \frac{1}{4} p^2$

Extract. sq. root, $x^{\frac{n}{2}} + \frac{1}{2} p = \pm \sqrt{q + \frac{1}{4} p^2}$

Transposing, $x^{\frac{n}{2}} = -\frac{1}{2} p \pm \sqrt{q + \frac{1}{4} p^2}$

By Note, $x = (-\frac{1}{2} p \pm \sqrt{q + \frac{1}{4} p^2})^{\frac{2}{n}}$

10. Given $\sqrt{x^5} - 3x = 40 x^{-\frac{1}{2}}$

Multiplying by $x^{\frac{1}{2}}$, $x^3 - 3 x^{\frac{3}{2}} = 40$

Mult. by 4, $4 x^3 - 12 x^{\frac{3}{2}} = 160$

Adding 3^2, $4 x^3 - 12 x^{\frac{3}{2}} + 9 = 169$

Extract. sq. root, $2 x^{\frac{3}{2}} - 3 = \pm 13$

Whence, $2 x^{\frac{3}{2}} = 3 \pm 13 = 16$, or -10

Dividing by 2, $x^{\frac{3}{2}} = 8$, or -5

By Note, $x = 8^{\frac{2}{3}}$, or $(-5)^{\frac{2}{3}}$

Or, $x = 4$, or $(-5)^{\frac{2}{3}}$

(Art. 278, pp. 236, 237.)

3. Given $(x-1)^2 - x = -\frac{3}{4}$

Adding 1, $(x-1)^2 - (x-1) = \frac{1}{4}$

Adding $(\frac{1}{2})^2$, $(x-1)^2 - (x-1) + \frac{1}{4} = 1$

Extr. sq. root, $(x-1) - \frac{1}{2} = \pm 1$

Or, $x - \frac{3}{2} = \pm 1$

Whence, $x = \frac{3}{2} \pm 1 = 2\frac{1}{2}$, or $\frac{1}{2}$

Note. If the given equation be expanded, it reduces to $x^2 - 3x = -\frac{5}{4}$, and this form, after completing the square and evolving, gives $x - \frac{3}{2} = \pm 1$, as above.

4. Given $(y^2 - 4y)^2 - 6(y^2 - 4y) + 5 = 0$

Trans. and add. $(\frac{6}{2})^2$, $(y^2 - 4y)^2 - 6(y^2 - 4y) + 9 = 4$

Extract. square root, $(y^2 - 4y) - 3 = \pm 2$

Whence, $y^2 - 4y = 3 \pm 2 = 5$, or 1

Adding $(\frac{4}{2})^2$, $y^2 - 4y + 4 = 9$, or 5

Extract. square root, $y - 2 = \pm 3$, or $\pm \sqrt{5}$

Transposing, $y = 2 \pm 3$, or $2 \pm \sqrt{5}$

Or, $y = 5$, or -1, or $2 \pm \sqrt{5}$

NOTE. If we substitute x^2 for $(y^2 - 4y)^2$, and x for $(y^2 - 4y)$, the original equation becomes, $x^2 - 6x = -5$, which gives the roots $x = 5$, or 1. Replacing the value of x, we have $y^2 - 4y = 5$, or 1, as before.

5. Given $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11$

Adding 5, $(x^2 - 2x + 5) + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 16$

Let $x^2 - 2x + 5 = y^2$, and $(x^2 - 2x + 5)^{\frac{1}{2}} = y$

Then, $y^2 + 6y = 16$

Adding $(\frac{6}{2})^2$, $y^2 + 6y + 9 = 25$

Extracting square root, $y + 3 = \pm 5$

Whence, $y = -3 \pm 5 = 2$, or -8

Squaring, $y^2 = 4$, or 64

Replac. val. of y^2, $x^2 - 2x + 5 = 4$, or 64

Transp. and uniting, $x^2 - 2x = -1$, or 59

Adding $(\frac{2}{2})^2$, $x^2 - 2x + 1 = 0$, or 60

Extracting square root, $x - 1 = 0$, or $\pm \sqrt{60}$

Transposing, $x = 1$, or $1 \pm \sqrt{60}$

Or, $x = 1$, or $1 \pm 2\sqrt{15}$

6. Given $\sqrt{x + 2} + 2\sqrt[4]{x + 2} = 8$

Adding $(\frac{2}{2})^2$, $\sqrt{x + 2} + 2\sqrt[4]{x + 2} + 1 = 9$

Extracting square root, $\sqrt[4]{x + 2} + 1 = \pm 3$

Whence, $\sqrt[4]{x + 2} = -1 \pm 3 = 2$, or -4

Raising to fourth power, $x + 2 = 16$, or 256

Transposing and uniting, $x = 14$, or 254

7. Given $(x^2 + 7)^{\frac{6}{5}} + 2(x^2 + 7)^{\frac{3}{5}} = 80$

Adding $(\frac{2}{2})^2$, $(x^2 + 7)^{\frac{6}{5}} + 2(x^2 + 7)^{\frac{3}{5}} + 1 = 81$

Extracting square root, $(x^2+7)^{\frac{3}{5}}+1=\pm 9$

Whence, $(x^2+7)^{\frac{3}{5}}=-1\pm 9=8, \text{ or } -10$

Extracting third root, $(x^2+7)^{\frac{1}{5}}=2, \text{ or } (-10)^{\frac{1}{3}}$

Raising to fifth power, $x^2+7=32, \text{ or } (-10)^{\frac{5}{3}}$

Transposing and uniting, $x^2=25, \text{ or } -7-10^{\frac{5}{3}}$

Extracting square root, $x=\pm 5$

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

(ART. 283, page 240.)

2. Given $\begin{cases} x+y=7 & (1) \\ x^2+2y^2=34 & (2) \end{cases}$

From (1), $x=7-y$ (3)

Squaring (3), $x^2=49-14y+y^2$ (4)

Subs. (4) in (2), $49-14y+y^2+2y^2=34$ (5)

Transposing and uniting, $3y^2-14y=-15$ (6)

Multiplying by 3, $9y^2-42y=-45$ (7)

Adding $(\frac{14}{2})^2$, $9y^2-42y+49=4$ (8)

Evolving, $3y-7=\pm 2$ (9)

Whence, $3y=9, \text{ or } 5$

Dividing by 3, $y=3, \text{ or } \frac{5}{3}$

Substituting in (3), $x=7-3, \text{ or } 7-\frac{5}{3}$

Uniting, $x=4, \text{ or } \frac{16}{3}$

3. Given $\begin{cases} x-\dfrac{x-y}{2}=4 & (1) \\ y-\dfrac{x+3y}{x+2}=1 & (2) \end{cases}$

Clearing (1) of fractions, $2x-x+y=8$ (3)

Clear. (2) of fract., $xy+2y-x-3y=x+2$ (4)

From (3), $y=8-x$ (5)

From (4), $xy-2x-y=2$ (6)

Subs. (5) in (6), $x(8-x)-2x-(8-x)=2$ (7)

Or, $8x - x^2 - 2x - 8 + x = 2$ (8)

Whence, $x^2 - 7x = -10$ (9)

Adding $(\frac{7}{2})^2$, $x^2 - 7x + \frac{49}{4} = \frac{9}{4}$ (10)

Evolving, $x - \frac{7}{2} = \pm \frac{3}{2}$ (11)

Whence, $x = 5$, or 2

Substituting in (5), $y = 8 - 5$, or $8 - 2$

Uniting, $y = 3$, or 6

4. Given $\begin{cases} x + 4y = 23 & (1) \\ x^2 + 3xy = 54 & (2) \end{cases}$

From (1), $y = \dfrac{23 - x}{4}$ (3)

Subs. (3) in (2), $x^2 + 3x\left(\dfrac{23 - x}{4}\right) = 54$ (4)

Exp. and mult. by 4, $4x^2 + 69x - 3x^2 = 216$ (5)

Uniting terms, $x^2 + 69x = 216$ (6)

Adding $(\frac{69}{2})^2$, $x^2 + 69x + \frac{4761}{4} = \frac{5625}{4}$ (7)

Evolving, $x + \frac{69}{2} = \pm \frac{75}{2}$ (8)

Whence, $x = 3$, or -72

Subs. in (3), $y = \dfrac{23 - 3}{4}$, or $\dfrac{23 + 72}{4}$

Whence, $y = 5$, or $\frac{95}{4}$

5. Given $\begin{cases} 49x^2 = 36y^2 & (1) \\ x(2x + \frac{1}{2}) + 3xy - y(6y + 5) + 128 = 0 & (2) \end{cases}$

Extracting square root of (1), $7x = 6y$ (3)

Expanding (2), $2x^2 + \frac{1}{2}x + 3xy - 6y^2 - 5y = -128$ (4)

From (3), $y = \dfrac{7x}{6}$ (5)

Subs. (5) in (4), $2x^2 + \frac{1}{2}x + \dfrac{7x^2}{2} - \dfrac{49x^2}{6} - \dfrac{35x}{6} = -128$ (6)

Mult. by 6, $12x^2 + 3x + 21x^2 - 49x^2 - 35x = -768$ (7)

Uniting terms, $-16x^2 - 32x = -768$ (8)

Dividing by -16, $x^2 + 2x = 48$ (9)

Adding $(\frac{2}{2})^2$, $x^2 + 2x + 1 = 49$ (10)

Evolving, $x + 1 = \pm 7$ (11)

Whence, $x = 6, \text{ or } -8$

Substituting in (5), $y = \frac{7 \times 6}{6}, \text{ or } \frac{-8 \times 7}{6}$

Whence, $y = 7, \text{ or } -\frac{28}{3}$

(ART. 284, page 241.)

2\. Given $\begin{cases} y^2 - x^2 = 3 & (1) \\ y^2 - 2xy + 2x^2 = 2 & (2) \end{cases}$

Let $y = vx$ (3)

Subs. in (1), $v^2x^2 - x^2 = 3$ (4)

Subs. in (2), $v^2x^2 - 2vx^2 + 2x^2 = 2$ (5)

From (4), $x^2 = \frac{3}{v^2 - 1}$ (6)

From (5), $x^2 = \frac{2}{v^2 - 2v + 2}$ (7)

Hence, $\frac{3}{v^2 - 1} = \frac{2}{v^2 - 2v + 2}$ (8)

Clear. of fractions, $3v^2 - 6v + 6 = 2v^2 - 2$ (9)

Transposing and uniting, $v^2 - 6v = -8$ (10)

Adding $(\frac{6}{2})^2$, $v^2 - 6v + 9 = 1$ (11)

Evolving, $v - 3 = \pm 1$ (12)

Whence, $v = 4, \text{ or } 2$

Substituting in (6), $x^2 = \frac{3}{16 - 1}, \text{ or } \frac{3}{4 - 1}$

Reducing, $x^2 = \frac{1}{5}, \text{ or } 1$

Evolving, $x = \pm \sqrt{\frac{1}{5}}, \text{ or } \pm 1$

Or, $x = \pm \frac{1}{5}\sqrt{5}, \text{ or } \pm 1$

Substituting in (3), $y = \pm \frac{1}{5}\sqrt{5} \times 4, \text{ or } \pm 1 \times 2$

Reducing, $y = \pm \frac{4}{5}\sqrt{5}, \text{ or } \pm 2$

3\. Given $\begin{cases} x^2 + 3xy - y^2 = 27 & (1) \\ 3x^2 + 2xy = 63 & (2) \end{cases}$

Let $y = vx$ (3)

Subs. in (1), $x^2 + 3vx^2 - v^2x^2 = 27$ (4)

Subs. in (2), $3x^2 + 2vx^2 = 63$ (5)

From (4), $x^2 = \dfrac{27}{1 + 3v - v^2}$ (6)

From (5), $x^2 = \dfrac{63}{3 + 2v}$ (7)

Hence, $\dfrac{27}{1 + 3v - v^2} = \dfrac{63}{3 + 2v}$ (8)

Dividing by 9, $\dfrac{3}{1 + 3v - v^2} = \dfrac{7}{3 + 2v}$ (9)

Clearing of fractions, $9 + 6v = 7 + 21v - 7v^2$ (10)

Tr. and uniting, $7v^2 - 15v = -2$ (11)

Mult. by 28, $196v^2 - 420v = -56$ (12)

Adding $(15)^2$, $196v^2 - 420v + 225 = 169$ (13)

Evolving, $14v - 15 = \pm 13$ (14)

Whence, $14v = 28, \text{ or } 2$ (15)

Dividing by 14, $v = 2, \text{ or } \frac{1}{7}$

Substituting in (7), $x^2 = \dfrac{63}{3 + 4}, \text{ or } \dfrac{63}{3 + \frac{2}{7}}$

Reducing, $x^2 = 9, \text{ or } \frac{441}{23}$

Evolving, $x = \pm 3, \text{ or } \pm \dfrac{21}{\sqrt{23}}$

Or, $x = \pm 3, \text{ or } \pm \frac{21}{23}\sqrt{23}$

Substituting in (3), $y = \pm 3 \times 2, \text{ or } \pm \frac{21}{23}\sqrt{23} \times \frac{1}{7}$

Reducing, $y = \pm 6, \text{ or } \pm \frac{3}{23}\sqrt{23}$

The last two examples may be solved in accordance with the principles contained in the note (Art. 284, Alg.), as follows:

2. Given $\begin{cases} y^2 - x^2 = 3 & (1) \\ y^2 - 2xy + 2x^2 = 2 & (2) \end{cases}$

From (1), $y^2 = 3 + x^2$ (3)

Evolving, $y = \pm\sqrt{3 + x^2}$ (4)

Subst. in (2), $3 + x^2 \mp 2x\sqrt{3 + x^2} + 2x^2 = 2$ (5)

Transp. and uniting, $3x^2 + 1 = \pm 2x\sqrt{3 + x^2}$ (6)

Squaring, $9x^4 + 6x^2 + 1 = 12x^2 + 4x^4$ (7)

Transp. and uniting, $5x^4 - 6x^2 = -1$ (8)

Multiplying by 5, $\quad 25x^4 - 30x^2 = -5 \quad (9)$

Adding $(\frac{6}{2})^2$, $\quad 25x^4 - 30x^2 + 9 = 4$

Evolving, $\quad 5x^2 - 3 = \pm 2$

Whence, $\quad 5x^2 = 5$, or 1

Dividing by 5, $\quad x^2 = 1$, or $\frac{1}{5}$

Evolving, $\quad x = \pm 1$, or $\pm \frac{1}{5}\sqrt{5}$

Substituting in (4), $\quad y = \pm\sqrt{3+1}$, or $\pm\sqrt{3+\frac{1}{5}}$

Reducing, $\quad y = \pm 2$, or $\pm \frac{4}{5}\sqrt{5}$

NOTE. Equation (8) is also readily obtained as follows: Subtract (2) from (1), and the result gives $y = \frac{3x^2+1}{2x}$, which, after squaring, may be substituted directly in (1).

3. Given $\left\{\begin{array}{l} x^2 + 3xy - y^2 = 27 \quad (1) \\ 3x^2 + 2xy = 63 \quad (2) \end{array}\right.$

From (2), $\quad y = \dfrac{63 - 3x^2}{2x} \quad (3)$

Squaring, $\quad y^2 = \dfrac{3969 - 378x^2 + 9x^4}{4x^2} \quad (4)$

Subs. in (1), $\quad x^2 + \dfrac{189 - 9x^2}{2} - \dfrac{3969 - 378x^2 + 9x^4}{4x^2} = 27 \quad (5)$

Clearing of fractions,

$4x^4 + 378x^2 - 18x^4 - 3969 + 378x^2 - 9x^4 = 108x^2 \quad (6)$

Whence, $\quad 23x^4 - 648x^2 = -3969 \quad (7)$

Multiplying by 23, $\quad 529x^4 - 14904x^2 = -91287 \quad (8)$

Adding $(\frac{648}{2})^2$, $\quad 529x^4 - 14904x^2 + (324)^2 = 13689$

Evolving, $\quad 23x^2 - 324 = \pm 117$

Whence, $\quad 23x^2 = 441$, or 207

Dividing by 23, $\quad x^2 = \frac{441}{23}$, or 9

Evolving, $\quad x = \pm \dfrac{21}{\sqrt{23}}$, or ± 3

Substituting in (3), $\quad y = \dfrac{63 - \frac{1323}{23}}{\pm \frac{42}{\sqrt{23}}}$, or $\dfrac{63 - 27}{\pm 6}$

Reducing, $\quad y = \pm \dfrac{3}{\sqrt{23}}$, or ± 6

NOTE. When both equations contain both powers of each letter, this method of elimination becomes rather tedious. In such a case we should first eliminate the square of one of the letters, and proceed as in the preceding note. It will be seen that y has but one power in one equation of each of the examples given.

(ART. 285, pp. 244, 245.)

4. Given $\begin{cases} x + y = 4 & (1) \\ x^{-1} + y^{-1} = 1 & (2) \end{cases}$

Multiplying (2) by xy,	$y + x = xy$	(3)
From (1) and (3),	$xy = 4$	(4)
Squaring (1),	$x^2 + 2xy + y^2 = 16$	(5)
Multiplying (4) by 4,	$4xy = 16$	(6)
Subtracting (6) from (5),	$x^2 - 2xy + y^2 = 0$	(7)
Evolving,	$x - y = 0$	(8)
Equation (1),	$x + y = 4$	
Adding (8) and (1),	$2x = 4$	
Whence,	$x = 2$	
Subtracting (8) from (1),	$2y = 4$	
Whence,	$y = 2$	

5. Given $\begin{cases} x^3 + y^3 = 65 & (1) \\ x + y = 5 & (2) \end{cases}$

Dividing (1) by (2),	$x^2 - xy + y^2 = 13$	(3)
Squaring (2),	$x^2 + 2xy + y^2 = 25$	(4)
Subtracting (3) from (4),	$3xy = 12$	(5)
Dividing (5) by 3,	$xy = 4$	(6)
Subtracting (6) from (3),	$x^2 - 2xy + y^2 = 9$	(7)
Evolving,	$x - y = \pm 3$	(8)
Equation (2),	$x + y = 5$	
Adding (8) and (2),	$2x = 8$, or 2	
Whence,	$x = 4$, or 1	
Subtracting (8) from (2),	$2y = 2$, or 8	
Whence,	$y = 1$, or 4	

6. Given
$$\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = 9 & (1) \\ x + y = 6 & (2) \end{cases}$$

Clearing (1) of fractions, $x^3 + y^3 = 9xy$ (3)

Dividing (3) by (2), $x^2 - xy + y^2 = \frac{3}{2}xy$ (4)

Transposing and uniting, $x^2 - \frac{5}{2}xy + y^2 = 0$ (5)

Squaring (2), $x^2 + 2xy + y^2 = 36$ (6)

Subtracting (5) from (6), $\frac{9}{2}xy = 36$ (7)

Whence, $xy = 8$ (8)

Multiplying by 4, $4xy = 32$ (9)

Subtracting (9) from (6), $x^2 - 2xy + y^2 = 4$ (10)

Evolving, $x - y = \pm 2$ (11)

Equation (2), $x + y = 6$

Adding (11) and (2), $2x = 8$, or 4

Whence, $x = 4$, or 2

Subtracting (11) from (2), $2y = 4$, or 8

Whence, $y = 2$, or 4

7. Given
$$\begin{cases} x^4 + x^2y^2 + y^4 = 931 & (1) \\ x^2 + xy + y^2 = 49 & (2) \end{cases}$$

Dividing (1) by (2), $x^2 - xy + y^2 = 19$ (3)

Subtracting (3) from (2), $2xy = 30$ (4)

Dividing by 2, $xy = 15$ (5)

Adding (5) to (2), $x^2 + 2xy + y^2 = 64$ (6)

Subtracting (5) from (3), $x^2 - 2xy + y^2 = 4$ (7)

Extracting square root of (6), $x + y = \pm 8$ (8)

Extracting square root of (7), $x - y = \pm 2$ (9)

Adding (9) and (8), $2x = \pm 10$, or ± 6

Whence, $x = \pm 5$, or ± 3

Subtracting (9) from (8), $2y = \pm 6$, or ± 10

Whence, $y = \pm 3$, or ± 5

NOTE. It will be seen that the original equations, after dividing one by the other, are homogeneous, and of the second degree; hence the ex-

ample might be classed under Case II. The double signs of the roots also indicate the nature of their equations. (Alg., Art. 284, Note.)

8. Given $\begin{cases} x + y = 61 & (1) \\ x^{\frac{1}{2}} + y^{\frac{1}{2}} = 11 & (2) \end{cases}$

FIRST SOLUTION.

Substituting v for $x^{\frac{1}{2}}$ and z for $y^{\frac{1}{2}}$, $\begin{cases} v^2 + z^2 = 61 & (3) \\ v + z = 11 & (4) \end{cases}$

Squaring (4), $v^2 + 2vz + z^2 = 121$ (5)
Subtracting (3) from (5), $2vz = 60$ (6)
Subtracting (6) from (3), $v^2 - 2vz + z^2 = 1$ (7)
Evolving, $v - z = \pm 1$ (8)
Equation (4), $v + z = 11$
Adding (8) and (4), $2v = 12$, or 10
Whence, $v = 6$, or 5
Subtracting (8) from (4), $2z = 10$, or 12
Whence, $z = 5$, or 6
Replacing $x^{\frac{1}{2}}$ and $y^{\frac{1}{2}}$, $x^{\frac{1}{2}} = 6$, or 5, and $y^{\frac{1}{2}} = 5$, or 6
Squaring, $x = 36$, or 25, and $y = 25$, or 36

SECOND SOLUTION.

Squaring (2), $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 121$ (3)
Equation (1), $x + y = 61$
Subtracting (1) from (3), $2x^{\frac{1}{2}}y^{\frac{1}{2}} = 60$ (4)
Squaring (4), $4xy = 3600$ (5)
Squaring (1), $x^2 + 2xy + y^2 = 3721$ (6)
Subtracting (5) from (6), $x^2 - 2xy + y^2 = 121$ (7)
Evolving, $x - y = \pm 11$ (8)
Equation (1), $x + y = 61$
Adding (8) and (1), $2x = 72$, or 50
Whence, $x = 36$, or 25
Subtracting (8) from (1), $2y = 50$, or 72
Whence, $y = 25$, or 36

(Art. 286, pp. 245, 246.)

1. Given $\begin{cases} x^2+y^2+xy-2x-2y=9 & (1) \\ xy=6 & (2) \end{cases}$

Adding (1) and (2), $(x+y)^2-2(x+y)=15$ (3)

Adding $(\frac{2}{2})^2$, $(x+y)^2-2(x+y)+1=16$ (4)

Evolving, $x+y-1=\pm 4$ (5)

Whence, $x+y=5, \text{ or } -3$ (6)

Squaring (6), $x^2+2xy+y^2=25$ (7)

Multiplying (2) by 4, $4xy \quad =24$ (8)

Subtracting (8) from (7), $x^2-2xy+y^2=1$ (9)

Evolving, $x-y=\pm 1$ (10)

Equation (6), $x+y=5$

Adding (10) and (6), $2x=6, \text{ or } 4$

Whence, $x=3, \text{ or } 2$

Subtracting (10) from (6), $2y=4, \text{ or } 6$

Whence, $y=2, \text{ or } 3$

2. Given $\begin{cases} 4xy=96-x^2y^2 & (1) \\ x+y=6 & (2) \end{cases}$

From (1), $x^2y^2+4xy+4=100$ (3)

Evolving, $xy+2=\pm 10$ (4)

Whence, $xy=8, \text{ or } -12$ (5)

Squaring (2), $x^2+2xy+y^2=36$ (6)

Multiplying (5) by 4, $4xy \quad =32, \text{ or } -48$ (7)

Subtracting (7) from (6), $x^2-2xy+y^2=4, \text{ or } 84$ (8)

Evolving, $x-y=\pm 2, \text{ or } \pm 2\sqrt{21}$ (9)

Equation (2), $x+y=6$

Adding (9) and (2), $2x=8, \text{ or } 4, \text{ or } 6\pm 2\sqrt{21}$

Whence, $x=4, \text{ or } 2, \text{ or } 3\pm\sqrt{21}$

Subtracting (9) from (2), $2y=4, \text{ or } 8, \text{ or } 6\mp 2\sqrt{21}$

Whence, $y=2, \text{ or } 4, \text{ or } 3\mp\sqrt{21}$

3\. Given $\begin{cases} \dfrac{x^2}{y^2} + \dfrac{4x}{y} = \dfrac{85}{9} & (1) \\ x - y = 2 & (2) \end{cases}$

Adding $\left(\frac{4}{2}\right)^2$ to (1), $\dfrac{x^2}{y^2} + \dfrac{4x}{y} + 4 = \dfrac{121}{9}$ (3)

Evolving, $\dfrac{x}{y} + 2 = \pm \dfrac{11}{3}$ (4)

Transposing, $\dfrac{x}{y} = -2 \pm \dfrac{11}{3}$ (5)

Uniting, $\dfrac{x}{y} = \dfrac{5}{3}$, or $-\dfrac{17}{3}$ (6)

Whence, $x = \dfrac{5y}{3}$, or $-\dfrac{17y}{3}$ (7)

Substituting in (2), $\dfrac{5y}{3} - y = 2$, or $-\dfrac{17y}{3} - y = 2$

Whence, $2y = 6$, or $-20y = 6$

Therefore, $y = 3$, or $-\frac{3}{10}$

Substituting in (7), $x = \dfrac{5 \times 3}{3}$, or $\left(-\dfrac{17}{3}\right) \times \left(-\dfrac{3}{10}\right)$

Reducing, $x = 5$, or $\frac{17}{10}$

(Art. 287, pp. 246, 247.)

1\. Given $\begin{cases} x^2 + xy = 60 & (1) \\ y^2 + xy = 84 & (2) \end{cases}$

Adding (1) and (2), $x^2 + 2xy + y^2 = 144$ (3)

Evolving, $x + y = \pm 12$ (4)

Dividing (1) by (4), $x = \pm 5$

Dividing (2) by (4), $y = \pm 7$

2\. Given $\begin{cases} x^2 + 9y^2 = 52 & (1) \\ x + 3y = 10 & (2) \end{cases}$

Squaring (2), $x^2 + 6xy + 9y^2 = 100$ (3)

Subtracting (1) from (3), $6xy = 48$ (4)

Subtracting (4) from (1), $x^2 - 6xy + 9y^2 = 4$ (5)

Evolving, $x - 3y = \pm 2$ (6)

Equation (2), $x + 3y = 10$

Adding (6) and (2), $2x = 12$, or 8

Whence, $x = 6$, or 4

Subtracting (6) from (2), $6y = 8$, or 12

Whence, $y = \frac{4}{3}$, or 2

NOTE. We take precisely the same steps that we should take if the equations were $x^2 + z^2 = 52$ and $x + z = 10$.

3. Given
$$\begin{cases} x^{\frac{2}{3}} + y^{\frac{1}{2}} = 7 & (1) \\ x^{\frac{4}{3}} y = 144 & (2) \end{cases}$$

Substituting v for $x^{\frac{2}{3}}$ and z for $y^{\frac{1}{2}}$,
$$\begin{cases} v + z = 7 & (3) \\ v^2 z^2 = 144 & (4) \end{cases}$$

Extracting square root of (4), $vz = \pm 12$ (5)

Squaring (3), $v^2 + 2vz + z^2 = 49$ (6)

Multiplying (5) by 4, $4vz = \pm 48$ (7)

Subtract. (7) from (6), $v^2 - 2vz + z^2 = 1$, or 97 (8)

Evolving, $v - z = \pm 1$, or $\pm \sqrt{97}$ (9)

Equation (3), $v + z = 7$

Adding (9) and (3), $2v = 8$, or 6, or $7 \pm \sqrt{97}$

Whence, $x^{\frac{2}{3}} = v = 4$, or 3, or $\frac{7 \pm \sqrt{97}}{2}$

Transposing exponent (Art. 277, Note),

$$x = 4^{\frac{3}{2}}, \text{ or } 3^{\frac{3}{2}}, \text{ or } \left(\frac{7 \pm \sqrt{97}}{2}\right)^{\frac{3}{2}}$$

Or, $x = \pm 8$, or $\pm 3\sqrt{3}$, or $\left(\frac{7 \pm \sqrt{97}}{2}\right)^{\frac{3}{2}}$

Subt. (9) from (3), $2z = 6$, or 8, or $7 \mp \sqrt{97}$

Whence, $y^{\frac{1}{2}} = z = 3$, or 4, or $\frac{7 \mp \sqrt{97}}{2}$

Squaring, $y = 9$, or 16, or $\left(\frac{7 \mp \sqrt{97}}{2}\right)^2$

The solution given above, expressed without the use of v and z, would be

Extracting square root of (2), $x^{\frac{2}{3}} y^{\frac{1}{2}} = \pm 12$ (3)

Squaring (1), $x^{\frac{4}{3}} + 2 x^{\frac{2}{3}} y^{\frac{1}{2}} + y = 49$ (4)

Multiplying (3) by 4, $4 x^{\frac{2}{3}} y^{\frac{1}{2}} = \pm 48$ (5)

Subt. (5) from (4), $x^{\frac{4}{3}} - 2 x^{\frac{2}{3}} y^{\frac{1}{2}} + y = 1$, or 97 (6)

Evolving, $x^{\frac{2}{3}} - y^{\frac{1}{2}} = \pm 1$, or $\pm \sqrt{97}$ (7)

Equation (1), $x^{\frac{2}{3}} + y^{\frac{1}{2}} = 7$

Adding (7) and (1), $2 x^{\frac{2}{3}} = 8$, or 6, or $7 \pm \sqrt{97}$

Subtracting (7) from (1), $2 y^{\frac{1}{2}} = 6$, or 8, or $7 \mp \sqrt{97}$

4. Given $\begin{cases} x^2 + 3xy = 54 & (1) \\ xy + 4y^2 = 115 & (2) \end{cases}$

Adding (1) and (2), $x^2 + 4xy + 4y^2 = 169$ (3)

Extracting square root of (3), $x + 2y = \pm 13$ (4)

From (4), $x = \pm 13 - 2y$ (5)

Subs. (5) in (2), $\pm 13y - 2y^2 + 4y^2 = 115$ (6)

Uniting terms, $2y^2 \pm 13y = 115$ (7)

Dividing by 2, $y^2 \pm \frac{13}{2} y = \frac{115}{2}$ (8)

Adding $(\frac{13}{4})^2$, $y^2 \pm \frac{13}{2} y + \frac{169}{16} = \frac{1089}{16}$ (9)

Evolving, $y \pm \frac{13}{4} = \pm \frac{33}{4}$ (10)

Transposing, $y = \mp \frac{13}{4} \pm \frac{33}{4}$ (11)

Uniting, $y = \pm 5$, or $\mp \frac{23}{2}$ (12)

Substituting in (5), $x = \pm 13 \mp 10$, or $\pm 13 \pm 23$ (13)

Uniting, $x = \pm 3$, or ± 36 (14)

NOTE. In equation (11), the signs $\mp$ and $\pm$ are entirely independent of each other, as they originate in different evolutions; hence they produce four distinct values of y, thus: $y = -\frac{13}{4} + \frac{33}{4}$, or $\frac{13}{4} - \frac{33}{4}$, or $-\frac{13}{4} - \frac{33}{4}$, or $\frac{13}{4} + \frac{33}{4}$. But the double signs of equation (13) are not all independent in their origin, as the sign of $2y$ is dependent upon that of 13; hence we do not here obtain four values from two double signs.

It is evident, from the form of the equations, and also of their roots, that the above example might be classed under Case II.

(ART. 288, pp. 247, 248.)

1. Given $\begin{cases} x^2 - y^2 = 24 & (1) \\ x + y = 6 & (2) \end{cases}$

Dividing (1) by (2),	$x - y = 4$	(3)
Adding (2) and (3),	$2x = 10$	(4)
Whence,	$x = 5$	(5)
Subtracting (3) from (2),	$2y = 2$	(6)
Whence,	$y = 1$	(7)

2. Given $\begin{cases} x + y = a & (1) \\ xy = b & (2) \end{cases}$

Squaring (1),	$x^2 + 2xy + y^2 = a^2$	(3)
Mult. (2) by 4,	$4xy \quad = 4b$	(4)
Subt. (4) from (3),	$x^2 - 2xy + y^2 = a^2 - 4b$	(5)
Evolving,	$x - y = \pm\sqrt{a^2 - 4b}$	(6)
Equation (1),	$x + y = a$	
Adding (6) and (1),	$2x = a \pm \sqrt{a^2 - 4b}$	
Whence,	$x = \dfrac{a \pm \sqrt{a^2 - 4b}}{2}$	
Subtracting (6) from (1),	$2y = a \mp \sqrt{a^2 - 4b}$	
Whence,	$y = \dfrac{a \mp \sqrt{a^2 - 4b}}{2}$	

3. Given $\begin{cases} x + y = a & (1) \\ x^2 + y^2 = c & (2) \end{cases}$

Squaring (1),	$x^2 + 2xy + y^2 = a^2$	(3)
Subt. (2) from (3),	$2xy = a^2 - c$	(4)
Subt. (4) from (2),	$x^2 - 2xy + y^2 = 2c - a^2$	(5)
Evolving,	$x - y = \pm\sqrt{2c - a^2}$	(6)
Equation (1),	$x + y = a$	
Adding (6) and (1),	$2x = a \pm \sqrt{2c - a^2}$	
Whence,	$x = \dfrac{a \pm \sqrt{2c - a^2}}{2}$	

Subtracting (6) from (1), $2y = a \mp \sqrt{2c - a^2}$

Whence, $y = \dfrac{a \mp \sqrt{2c - a^2}}{2}$

NOTE. Equation (5) may also be obtained by subtracting (3) from (2) multiplied by 2.

4. Given $\begin{cases} x + y = 2 & (1) \\ x^{-1} + y^{-1} = 2 & (2) \end{cases}$

Multiplying (2) by xy, $y + x = 2xy$ (3)
From (1) and (3), $2xy = 2$ (4)
Squaring (1), $x^2 + 2xy + y^2 = 4$ (5)
Multiplying (4) by 2, $4xy \quad = 4$ (6)
Subtracting (6) from (5), $x^2 - 2xy + y^2 = 0$ (7)
Evolving, $x - y = 0$ (8)
Equation (1), $x + y = 2$
Adding (8) and (1), $2x = 2$
Whence, $x = 1$
Subtracting (8) from (1), $2y = 2$
Whence, $y = 1$

5. Given $\begin{cases} x^{-2} + y^{-2} = \frac{5}{9} & (1) \\ x^{-1} + y^{-1} = 1 & (2) \end{cases}$

Squaring (2), $x^{-2} + 2x^{-1}y^{-1} + y^{-2} = 1$ (3)
Subtracting (1) from (3), $2x^{-1}y^{-1} = \frac{4}{9}$ (4)
Subtracting (4) from (1), $x^{-2} - 2x^{-1}y^{-1} + y^{-2} = \frac{1}{9}$ (5)
Evolving, $x^{-1} - y^{-1} = \pm\frac{1}{3}$ (6)
Equation (2), $x^{-1} + y^{-1} = 1$
Adding (6) and (2), $2x^{-1} = \frac{4}{3}$, or $\frac{2}{3}$
Dividing by 2, $x^{-1} = \frac{2}{3}$, or $\frac{1}{3}$
Whence, $x = \frac{3}{2}$, or 3
Subtracting (6) from (2), $2y^{-1} = \frac{2}{3}$, or $\frac{4}{3}$
Dividing by 2, $y^{-1} = \frac{1}{3}$, or $\frac{2}{3}$
Whence, $y = 3$, or $\frac{3}{2}$

NOTE. If it is preferred to do so, the negative exponents may be re

moved at once, transforming (1) and (2) to $x^2 + y^2 = \frac{5x^2y^2}{9}$ and $x + y = xy$, from which $x + y = \frac{9}{2}$ and $xy = \frac{9}{2}$ are easily obtained.

6. Given $\begin{cases} x^3 - y^3 = 8 & (1) \\ x - y = 2 & (2) \end{cases}$

Dividing (1) by (2), $x^2 + xy + y^2 = 4$ (3)

Squaring (2), $x^2 - 2xy + y^2 = 4$ (4)

Subtracting (4) from (3), $3xy = 0$ (5)

Whence, $xy = 0$ (6)

Adding (6) to (3), $x^2 + 2xy + y^2 = 4$ (7)

Evolving, $x + y = \pm 2$ (8)

Equation (2), $x - y = 2$

Adding (8) and (2), $2x = 4$, or 0

Whence, $x = 2$, or 0

Subtracting (2) from (8), $2y = 0$, or -4

Whence, $y = 0$, or -2

7. Given $\begin{cases} x^4 + y^4 = 82 & (1) \\ xy = 3 & (2) \end{cases}$

Squaring (2), $x^2y^2 = 9$ (3)

Multiplying (3) by (2), $2x^2y^2 = 18$ (4)

Adding (4) to (1), $x^4 + 2x^2y^2 + y^4 = 100$ (5)

Subtracting (4) from (1), $x^4 - 2x^2y^2 + y^4 = 64$ (6)

Extracting square root of (5), $x^2 + y^2 = \pm 10$ (7)

Extracting square root of (6), $x^2 - y^2 = \pm 8$ (8)

Adding (7) and (8), $2x^2 = \pm 18$, or ± 2

Dividing by 2, $x^2 = \pm 9$, or ± 1

Evolving, $x = \pm 3$, or ± 1, or $\pm 3\sqrt{-1}$, or $\pm\sqrt{-1}$

Subtracting (8) from (7), $2y^2 = \pm 2$, or ± 18

Dividing by 2, $y^2 = \pm 1$, or ± 9

Evolving, $y = \pm 1$, or ± 3, or $\pm\sqrt{-1}$, or $\pm 3\sqrt{-1}$

NOTE. The same results may be obtained from (2) and (7), as in Ex. 2, Art. 285, without the use of (6) and (8).

8. Given $\begin{cases} x - y = 8(\sqrt{x} - \sqrt{y}) & (1) \\ \sqrt{xy} = 15 & (2) \end{cases}$

$$
\begin{array}{lrl}
\text{Dividing (1) by } \sqrt{x}-\sqrt{y}, & \sqrt{x}+\sqrt{y} = 8 & (3) \\
\text{Squaring (3),} & x + 2\sqrt{xy} + y = 64 & (4) \\
\text{Substituting (2) in (4),} & x + y = 34 & (5) \\
\text{Squaring (2),} & xy = 225 & (6) \\
\text{Squaring (5),} & x^2 + 2xy + y^2 = 1156 & (7) \\
\text{Multiplying (6) by 4,} & 4xy \quad = \ 900 & (8) \\
\text{Subt. (8) from (7),} & x^2 - 2xy + y^2 = \ 256 & (9) \\
\text{Evolving,} & x - y = \pm 16 & (10) \\
\text{Equation (5),} & x + y = 34 & \\
\text{Adding (10) and (5),} & 2x = 50, \text{ or } 18 & \\
\text{Whence,} & x = 25, \text{ or } 9 & \\
\text{Subtracting (10) from (5),} & 2y = 18, \text{ or } 50 & \\
\text{Whence,} & y = 9, \text{ or } 25 &
\end{array}
$$

NOTE. After obtaining equation (4), we might subtract from it four times equation (2), and the square root of the result would be $\sqrt{x} - \sqrt{y} = \pm 2$. This, combined with (3), gives $\sqrt{x} = 5$, or 3, and $\sqrt{y} = 3$, or 5, whence $x = 25$, or 9, and $y = 9$, or 25, as above.

Since we divide (1) by $\sqrt{x} - \sqrt{y}$, it is evident that $\sqrt{x} - \sqrt{y} = 0$, or $x = y$, will satisfy that equation. This, combined with (2), gives the additional roots of the equations, $x = 15$, $y = 15$. The same results may be obtained by eliminating one of the unknown quantities from (1) and (2) by substitution, without dividing by $\sqrt{x} - \sqrt{y}$. The resulting equation may be put in the form

$$x^2 - 8x^{\frac{3}{2}} + 16x = 16x - 120x^{\frac{1}{2}} + (15)^2$$

which gives $\quad x - 4x^{\frac{1}{2}} = \pm(4x^{\frac{1}{2}} - 15)$

whence $\quad x = 25$, or 9, or 15

Also $\quad y = 9$, or 25, or 15

The forms of the roots show that equation (1) is *symmetrical* in signs, as well as in other respects. (Art. 285, Ex. 1.) This may also be inferred from the fact that if all its signs be changed, the signs of x and y will be reversed. Equation (1) is also changed to the symmetrical form $x - 8\sqrt{x} = y - 8\sqrt{y}$ by simply transposing.

9. Given $\begin{cases} x+y=72 & (1) \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=6 & (2) \end{cases}$

Substituting v for $x^{\frac{1}{3}}$ and z for $y^{\frac{1}{3}}$, $\begin{cases} v^3+z^3=72 & (3) \\ v+z=6 & (4) \end{cases}$

Dividing (3) by (4), $v^2-vz+z^2=12 \quad (5)$

Squaring (4), $v^2+2vz+z^2=36 \quad (6)$

Subtracting (5) from (6), $3vz=24 \quad (7)$

Whence, $vz=8 \quad (8)$

Subtracting (8) from (5), $v^2-2vz+z^2=4 \quad (9)$

Evolving, $v-z=\pm 2 \quad (10)$

Equation (4), $v+z=6$

Adding (10) and (4), $2v=8$, or 4

Dividing by 2, $v=4$, or 2

Whence, $x=v^3=64$, or 8

Subtracting (10) from (4), $2z=4$, or 8

Dividing by 2, $z=2$, or 4

Whence, $y=z^3=8$, or 64

NOTE. The value of $x^{\frac{1}{3}}y^{\frac{1}{3}}$ may be found, just as equation (8) is obtained, without the use of v and z. The remaining portion of the solution may be made to correspond exactly with the above, x and y with fractional exponents being used in place of v and z; but, if it is preferred, $x^{\frac{1}{3}}y^{\frac{1}{3}}=8$ may be cubed, and we then have $x+y=72$ and $xy=512$, from which x and y can be obtained without any further use of fractional exponents. In the latter case we must work with larger numbers, but we avoid fractional exponents; in the former we have the advantage of smaller numbers, but we must work with fractional exponents.

One of the unknown quantities may be eliminated by substitution in either of the preceding examples, and the resulting equation will be a quadratic, or in the quadratic form, except in the first example.

10. Given $\begin{cases} x+y:x-y::13:5 & (1) \\ y^2+x=25 & (2) \end{cases}$

From (1), $5x+5y=13x-13y \quad (3)$

Or, $18y=8x \quad (4)$

Whence, $x = \frac{9y}{4}$ (5)

Substituting in (2), $y^2 + \frac{9y}{4} = 25$ (6)

Complet. the square, $y^2 + \frac{9y}{4} + \frac{81}{64} = \frac{1681}{64}$ (7)

Evolving, $y + \frac{9}{8} = \pm \frac{41}{8}$

Whence, $y = 4$, or $-6\frac{1}{4}$

Substituting in (5), $x = \frac{9 \times 4}{4}$, or $-\frac{25}{4} \times \frac{9}{4}$

Reducing, $x = 9$, or $-14\frac{1}{16}$

NOTE. As these equations are not symmetrical, the only proper course is to eliminate one of the unknown quantities at once, and solve the quadratic equation thus produced according to the principles of Case I.

11. Given $\begin{cases} x + 4y = 14 & (1) \\ 4x - 2y + y^2 = 11 & (2) \end{cases}$

From (1), $x = 14 - 4y$ (3)

Subst. in (2), $56 - 16y - 2y + y^2 = 11$ (4)

Or, $y^2 - 18y = -45$ (5)

Completing square, $y^2 - 18y + 81 = 36$ (6)

Evolving, $y - 9 = \pm 6$

Whence, $y = 15$, or 3

Substituting in (3), $x = 14 - 60$, or $14 - 12$

Uniting, $x = -46$, or 2

12. Given $\begin{cases} x + 3y = 7 & (1) \\ x^2 - 3xy + 3y^2 = 7 & (2) \end{cases}$

From (1), $x = 7 - 3y$ (3)

Subs. in (2), $49 - 42y + 9y^2 - 21y + 9y^2 + 3y^2 = 7$ (4)

Whence, $21y^2 - 63y = -42$ (5)

Dividing by 21, $y^2 - 3y = -2$ (6)

Completing the square, $y^2 - 3y + \frac{9}{4} = \frac{1}{4}$ (7)

Evolving, $y - \frac{3}{2} = \pm \frac{1}{2}$

Whence, $y = 2$, or 1

Substituting in (3), $x = 7 - 6$, or $7 - 3$

Uniting, $x = 1$, or 4

THEORY OF QUADRATIC EQUATIONS.

(ART. 290, page 250.)

1. Given $x^2 - 4x + 3 = 0$
Transp. and compl. square, $x^2 - 4x + 4 = 1$
Evolving, $x - 2 = \pm 1$
Whence, $x = 3$, or 1
Hence, $x^2 - 4x + 3 = (x - 3)(x - 1) = 0$

2 Given $x^2 - \frac{x}{6} - \frac{1}{3} = 0$
Transp. and compl. square, $x^2 - \frac{x}{6} + \frac{1}{144} = \frac{49}{144}$
Evolving, $x - \frac{1}{12} = \pm \frac{7}{12}$
Whence, $x = \frac{2}{3}$, or $-\frac{1}{2}$
Hence, $x^2 - \frac{x}{6} - \frac{1}{3} = (x - \frac{2}{3})(x + \frac{1}{2}) = 0$

3. Given $x^2 - 7x + 12 = 0$
Transp. and compl. square, $x^2 - 7x + \frac{49}{4} = \frac{1}{4}$
Evolving, $x - \frac{7}{2} = \pm \frac{1}{2}$
Whence, $x = 4$, or 3
Ans. $(x - 4)(x - 3) = 0$

4. Given $x^2 + 6x + 8 = 0$
Transp. and compl. square, $x^2 + 6x + 9 = 1$
Evolving, $x + 3 = \pm 1$
Whence, $x = -2$, or -4
Ans. $(x + 2)(x + 4) = 0$

NOTE. The principles established in Art. 289 will determine whether the roots obtained for any quadratic equation are correct; and in some cases they will enable us to obtain those roots by inspection. Thus, in the third example, it is evident that 4 and 3 are the only numbers whose sum is 7, and whose product is 12.

(ART. 291, page 251.)

1. Assume $x^2 - 5x + 6 = 0$. It is evident that 3 and 2 are the only numbers whose sum is 5 and whose product is 6; hence they are the roots of the equation, and

$$x^2 - 5x + 6 = (x - 3)(x - 2).$$

2. Assume $\quad x^2 - \frac{x}{12} - \frac{5}{8} = 0$

Completing square, $\quad x^2 - \frac{x}{12} + \frac{1}{576} = \frac{361}{576}$

Evolving, $\quad x - \frac{1}{24} = \pm\, \frac{19}{24}$

Whence, $\quad x = \frac{5}{6}$, or $-\frac{3}{4}$

Therefore, $x^2 - \frac{x}{12} - \frac{5}{8} = (x - \frac{5}{6})(x + \frac{3}{4})$

3. Assume, $\quad x^2 + 3x - 28 = 0$

Completing square, $\quad x^2 + 3x + \frac{9}{4} = \frac{121}{4}$

Evolving, $\quad x + \frac{3}{2} = \pm\, \frac{11}{2}$

Whence, $\quad x = 4$, or -7

$x^2 + 3x - 28 = (x - 4)(x + 7)$, Ans.

FORMATION OF EQUATIONS.

(ART. 292, page 251.)

2. $(x - 4)(x - 5) = x^2 - 9x + 20 = 0$

Or, $\quad x^2 - 9x = -20$

3. $(x - 6)(x - 7) = x^2 - 13x + 42 = 0$

Or, $\quad x^2 - 13x = -42$, Ans.

4. $(x + 1)(x + 2) = x^2 + 3x + 2 = 0$

Or, $\quad x^2 + 3x = -2$

PROBLEMS

LEADING TO AFFECTED QUADRATIC EQUATIONS.

(Art. 299, pp. 253-259.)

2. Let $x =$ the number.

Then, $$x - \left(\frac{x}{4}\right)^2 = 3$$

Or, $$x^2 - 16x = -48$$

Completing square, $$x^2 - 16x + 64 = 16$$

Evolving, $$x - 8 = \pm 4$$

Whence, $$x = 12, \text{ or } 4$$

Verification. $$\begin{cases} 12 - 3^2 = 3 \\ 4 - 1^2 = 3 \end{cases}$$

5. Let $x =$ the cost.

Then $x =$ the gain per cent.,

and $\frac{x}{100} \times x = \frac{x^2}{100} =$ the whole gain.

Therefore, $$x + \frac{x^2}{100} = 56$$

Or, $$x^2 + 100x = 5600$$

Completing square, $$x^2 + 100x + 2500 = 8100$$

Evolving, $$x + 50 = \pm 90$$

Whence, $$x = 40, \text{ or } -140$$

6. Let $x =$ the number of chickens.

Then $\frac{96}{x} =$ cost of one chicken,

and $\frac{27x}{2} =$ am't rec'd for the chickens

Therefore, $$\frac{27x}{2} - 96 = \frac{96}{x}$$

Clearing of fractions, $$27x^2 - 192x = 192$$

Dividing by 27, $$x^2 - \tfrac{64}{9}x = \tfrac{64}{9}$$

Completing square, $$x^2 - \tfrac{64}{9}x + (\tfrac{32}{9})^2 = \tfrac{1600}{81}$$

Evolving, $$x - \tfrac{32}{9} = \pm \tfrac{40}{9}$$

Whence, $$x = 8, \text{ or } -\tfrac{8}{9}$$

7. Let $x =$ the number of pages.

Then $x - 5 =$ estimated no. pages,

$\frac{80}{x - 5} =$ estimated cost per page,

and $\frac{67.50}{x} = \frac{135}{2x} =$ am't received per page.

Therefore, $$\frac{80}{x-5} - \frac{135}{2x} = \frac{1}{2}$$

Clearing of fractions, $160x - 135x + 675 = x^2 - 5x$

Whence, $x^2 - 30x = 675$

Completing square, $x^2 - 30x + 225 = 900$

Evolving, $x - 15 = \pm 30$

Whence, $x = 45$, or -15

If "5 pages more" be changed to "5 pages *less*," and "abatement" to "*additional charge*," then 15 will be the true number of pages.

8. Let $x =$ no. persons at first,

and $x - 2 =$ no. persons who paid.

Then $\frac{8.75}{x} = \frac{35}{4x} =$ am't each should have paid,

and $\frac{8.75}{x-2} = \frac{35}{4x-8} =$ am't each did pay.

Therefore, $$\frac{35}{4x-8} - \frac{35}{4x} = \frac{1}{2}$$

Clearing of fractions, $140x - 140x + 280 = 8x^2 - 16x$

Whence, $8x^2 - 16x = 280$

Dividing by 8, $x^2 - 2x = 35$

Completing square, $x^2 - 2x + 1 = 36$

Evolving, $x - 1 = \pm 6$

Whence, $x = 7$, or -5

The number 5 is the answer to a problem like the above, except that two *additional* persons came, and each of the company had therefore 50 cts. *less* to pay.

9. Let $x =$ original no. persons,

and $x + 2 =$ whole no. persons.

Then $\frac{1000}{x}$ = sum each expected originally,

and $\frac{1000}{x+2}$ = sum each will receive.

Therefore, $\frac{1000}{x} - \frac{1000}{x+2} = 25$

Clear. of fractions, $1000x + 2000 - 1000x = 25x^2 + 50x$

Or, $25x^2 + 50x = 2000$

Dividing by 25, $x^2 + 2x = 80$

Completing square, $x^2 + 2x + 1 = 81$

Evolving, $x + 1 = \pm 9$

Whence, $x = 8$, or -10

The number 10 is the answer to a problem similar to the one given, except that two claimants *disappear*, and each receives $ 25 *more* than he expected.

10. Let x = width of frame.

Then $18 + 2x$ = outside length of frame,

and $12 + 2x$ = outside width of frame.

Also $18 \times 12 = 216$ = surface of glass,

and $(18 + 2x)(12 + 2x)$ = whole surface.

Therefore, $(18 + 2x)(12 + 2x) = 2 \times 216$

Expanding, $216 + 60x + 4x^2 = 432$

Or, $4x^2 + 60x = 216$

Dividing by 4, $x^2 + 15x = 54$

Completing square, $x^2 + 15x + \frac{225}{4} = \frac{441}{4}$

Evolving, $x + \frac{15}{2} = \pm \frac{21}{2}$

Whence, $x = 3$, or -18

NOTE. The equation $4x^2 + 60x = 216$ may be obtained at once by estimating the surface of the frame. $4x^2$ is the area of the four corners; there are two sides each 18 by x, and two ends each 12 by x; hence the whole surface of the frame is $4x^2 + 60x$, which, according to the conditions of the problem, is equal to the surface of the glass.

12. FIRST SOLUTION.

Let x = one part,

and y = the other part.

Then,	$x+y=40$	(1)
and	$x^2+y^2=818$	(2)
Squaring (1),	$x^2+2xy+y^2=1600$	(3)
Subtracting (2) from (3),	$2xy=782$	(4)
Subtr. (4) from (2),	$x^2-2xy+y^2=36$	(5)
Evolving,	$x-y=\pm 6$	(6)
Equation (1),	$x+y=40$	
Adding (6) and (1),	$2x=46$, or 34	
Whence,	$x=23$, or 17	
Subtracting (6) from (1),	$2y=34$, or 46	
Whence,	$y=17$, or 23	

NOTE. Either letter may also be eliminated by substitution, or only one unknown quantity may be used, if preferred.

SECOND SOLUTION.

Let	$x-y=$ one part,	
and	$x+y=$ the other part.	
Their sum,	$2x=40$	(1)
The sum of their squares,	$2x^2+2y^2=818$	(2)
From (1),	$x=20$	(3)
Squaring,	$x^2=400$	(4)
From (2),	$x^2+y^2=409$	(5)
Subtracting (4) from (5),	$y^2=9$	(6)
Evolving,	$y=\pm 3$	(7)
From (3) and (7),	$x-y=17$, or 23	
Also,	$x+y=23$, or 17	

13. Let $x=$ one part,
and $y=$ the other part.

Then,	$x+y=60$	(1)
and	$xy:x^2+y^2::2:5$	(2)
From (2),	$2x^2+2y^2=5xy$	(3)
Or,	$x^2-\frac{5}{2}xy+y^2=0$	(4)
Squaring (1),	$x^2+2xy+y^2=3600$	(5)

Subtracting (4) from (5), $\frac{9}{2}xy = 3600$ (6)

Whence, $xy = 800$ (7)

Or, $4xy = 3200$ (8)

Subtracting (8) from (5), $x^2 - 2xy + y^2 = 400$ (9)

Evolving, $x - y = \pm 20$ (10)

Equation (1), $x + y = 60$

Adding (10) and (1), $2x = 80$, or 40

Whence, $x = 40$, or 20

Subtracting (10) from (1), $2y = 40$, or 80

Whence, $y = 20$, or 40

NOTE. This problem may be solved by the various methods indicated in connection with the solutions of the last example. If $x - y$ and $x + y$ are used, the equations become $2x = 60$ and $x^2 - y^2 : 2x^2 + 2y^2 :: 2 : 5$, or $x = 30$ and $x^2 = 9y^2$, from which $y = \pm 10$, and the numbers are 20 and 40, as before.

15. Let $x =$ cost of 1st per yard, in doll.,

and $y =$ cost of 2d per yard, in doll.

Then $100y =$ gain per cent of first,

and $100y =$ loss per cent of second.

Also $x \times \frac{100y}{100} = xy =$ gain on a yard of first,

and $y \times \frac{100y}{100} = y^2 =$ loss on a yard of second.

Therefore, $80x + 80y = 60$ (1)

Also, $80xy - 80y^2 = 5$ (2)

From (1), $4x + 4y = 3$ (3)

From (2), $16xy - 16y^2 = 1$ (4)

Mult. (3) by $4y$, $16xy + 16y^2 = 12y$ (5)

Subt. (4) from (5), $32y^2 = 12y - 1$ (6)

Whence, $y^2 - \frac{3}{8}y = -\frac{1}{32}$ (7)

Completing square, $y^2 - \frac{3}{8}y + \frac{9}{256} = \frac{1}{256}$ (8)

Evolving, $y - \frac{3}{16} = \pm\frac{1}{16}$ (9)

Whence, $y = \frac{1}{4}$, or $\frac{1}{8}$

From (3), $x = \frac{3-4y}{4} = \frac{3-1}{4}$, or $\frac{3-\frac{1}{2}}{4}$

Reducing, $x = \frac{1}{2}$, or $\frac{5}{8}$

It is evident that $\$\frac{1}{2} = 50$ cts., $\$\frac{5}{8} = 62\frac{1}{2}$ cts., $\$\frac{1}{4} =$ 25 cts., and $\$\frac{1}{8} = 12\frac{1}{2}$ cts.

NOTE. If x and y represent the cost of each in *cents*, then y will represent the gain and loss per cent, and the equations will become

$$80x + 80y = 6000, \qquad \text{and } \frac{80xy}{100} - \frac{80y^2}{100} = 500,$$

whence $x + y = 75$, and $xy - y^2 = 625$.
These give $2y^2 - 75y = -625$, $y = 25$, or $12\frac{1}{2}$, and $x = 50$, or $62\frac{1}{2}$.

16. Let $x =$ the greater,
and $y =$ the less.

Then, $x(x+y) = 144$ (1)
and $y(x-y) = 14$ (2)

Expanding (1), $x^2 + xy = 144$ (3)
Expanding (2), $xy - y^2 = 14$ (4)

From (4), $x = \frac{14+y^2}{y}$ (5)

Subst. in (3), $\frac{196+28y^2+y^4}{y^2} + 14 + y^2 = 144$ (6)

Cl. of fract., $196 + 28y^2 + y^4 + 14y^2 + y^4 = 144y^2$ (7)
Whence, $2y^4 - 102y^2 = -196$ (8)
Multiplying by 2, $4y^4 - 204y^2 = -392$ (9)
Adding $(\frac{102}{2})^2$, $4y^4 - 204y^2 + 2601 = 2209$ (10)
Evolving, $2y^2 - 51 = \pm 47$ (11)
Whence, $2y^2 = 98$, or 4
Dividing by 2, $y^2 = 49$, or 2
Evolving, $y = \pm 7$, or $\pm\sqrt{2}$

Substituting in (5), $x = \frac{14+49}{\pm 7}$, or $\frac{14+2}{\pm\sqrt{2}}$

Reducing, $x = \pm 9$, or $\pm 8\sqrt{2}$

NOTE. As both the equations are homogeneous and of the second degree, they may be solved by assuming $y = vx$, according to the method of Art. 284.

17. Let $x =$ no. bushels.

Then $\frac{60}{x} =$ price paid per bush., in dollars,

and $\frac{54}{x-15} =$ price rec'd per bush., in dollars.

Therefore, $$\frac{54}{x-15} - \frac{60}{x} = \frac{1}{10}$$

Clearing of fractions, $540x - 600x + 9000 = x^2 - 15x$

Whence, $x^2 + 45x = 9000$

Completing square, $x^2 + 45x + (\frac{45}{2})^2 = \frac{38025}{4}$

Evolving, $x + \frac{45}{2} = \pm \frac{195}{2}$

Whence, $x = 75, \text{ or } -120$

If he had *added* 15 bushels, instead of reserving 15, and *lost* 10 cents a bushel by selling for $ 54, then 120 bushels would be the true answer.

18. Let $x =$ first number of sheep,

and $y =$ second number of sheep.

Then $\frac{80}{y} =$ price of first, per sheep,

and $\frac{180}{x} =$ price of second, per sheep.

Also $\frac{80x}{y} =$ sum received for first,

and $\frac{180y}{x} =$ sum received for second.

Then, $x + y = 100 \quad (1)$

and $\frac{180y}{x} = \frac{80x}{y} \quad (2)$

Clearing (2) of fractions, $180y^2 = 80x^2 \quad (3)$

Dividing by 20, $9y^2 = 4x^2 \quad (4)$

Evolving, $3y = \pm 2x \quad (5)$

Dividing by 3, $y = \pm \frac{2x}{3} \quad (6)$

Substituting (6) in (1), $x \pm \frac{2x}{3} = 100 \quad (7)$

Clearing of fractions, $3x \pm 2x = 300$ (8)

Whence, $x = \frac{300}{3 \pm 2} = 60$, or 300

Subst. in (1), $y = 100 - x = 100 - 60$, or $100 - 300$

Uniting, $y = 40$, or -200

Then, $\frac{80}{y} = \frac{80}{40} = 2$

Also, $\frac{180}{x} = \frac{180}{60} = 3$

RATIO AND PROPORTION.

(Art. 329, pp. 268, 269.)

3. Let $x =$ first term.

Then, $x : 4 :: 6 : 8$

By Theo. XI., $x : 4 :: 3 : 4$

By Theo. XII., $x : 1 :: 3 : 1$

By Theo. I., $x = 3$, Ans.

Note. We may, of course, apply Theo. I. to the first proportion, thus producing the equation $8x = 24$, which may be solved in the usual manner.

4. Given $3 : x :: x : 1083$

By Theo. I., $x^2 = 3249$

Evolving, $x = \pm 57$

5. Given $a + x : a - x :: 11 : 7$

By Theo. VIII., $2a : 2x :: 18 : 4$

Dividing by 2, (Theo. XI.), $a : x :: 9 : 2$

6. Let A and B represent the altitude and base of one triangle, and a and b the altitude and base of the other.

Then, $B : b :: 17 : 18$

and $A : a :: 21 : 23$

By Theo. XIII., $B \times A : b \times a :: 17 \times 21 : 18 \times 23$

Or, $B \times A : b \times a :: 357 : 414$

Div. by 3, (Theo. XI.), $B \times A : b \times a :: 119 : 138$

7. Let $x =$ no. gallons of milk.

Then $\frac{5x}{4} =$ no. gallons of mixture at first,

$\frac{5x}{4} - 3 =$ no. gallons mixture after selling,

and $\frac{5x}{4} - 3 + \frac{3}{4} = \frac{5x-9}{4} =$ no. gals. mixt. at last.

Therefore, $\frac{5x-9}{4} : \frac{5x-12}{4} :: 7 : 6$

Or (Theo. XI.), $5x - 9 : 5x - 12 :: 7 : 6$

By Theo. VII., $3 : 5x - 12 :: 1 : 6$

By Theo. I., $5x - 12 = 18$

Transposing and uniting, $5x = 30$

Dividing by 5, $x = 6$

8. Let $x =$ A's stock,

and $y =$ B's stock.

Then, $x + 100 : y - 50 :: 4 : 3$ (1)

and $x - 50 : y + 100 :: 5 : 9$ (2)

From (1) (Theo. I.), $3x + 300 = 4y - 200$ (3)

From (2) (Theo. I.), $9x - 450 = 5y + 500$ (4)

Transp. and unit. in (3), $3x - 4y = -500$ (5)

Transp. and unit. in (4), $9x - 5y = 950$ (6)

Multiplying (5) by 3, $9x - 12y = -1500$ (7)

Subtracting (7) from (6), $7y = 2450$ (8)

Dividing by 7, $y = 350$ (9)

Substituting in (5), $3x - 1400 = -500$ (10)

Transp. and uniting, $3x = 900$ (11)

Dividing by 3, $x = 300$ (12)

9. Let x and y represent the numbers.

Then, $xy = 15$ (1)

and $x^2 + y^2 : x^2 - y^2 :: 17 : 8$ (2)

From (2), (Theo. VIII.), $2x^2 : 2y^2 :: 25 : 9$ (3)

Dividing by 2, (Theo. XI.), $x^2 : y^2 :: 25 : 9$ (4)

Evolving (Theo. XIV.), $x : y :: 5 : 3$ (5)

Whence (Theo. I.), $3x = 5y$ (6)

Dividing by 3, $x = \frac{5y}{3}$ (7)

Substituting (7) in (1), $\frac{5y^2}{3} = 15$ (8)

Whence, $y^2 = 9$ (9)

Evolving, $y = \pm 3$ (10)

Substituting in (7), $x = \frac{\pm 3 \times 5}{3} = \pm 5$ (11)

10. Let $x =$ A's money
$y =$ B's money,

and $r =$ ratio of the money staked to the whole amount of each.

Then $rx =$ money staked by A,
and $ry =$ money staked by B.

Therefore, $x + ry = 2(y - ry)$ (1)

$y + rx = 3(x - rx)$ (2)

and $x + y = 168$ (3)

From (1), $x + 3ry = 2y$ (4)

From (2), $y + 4rx = 3x$ (5)

From (4), $r = \frac{2y - x}{3y}$ (6)

From (5), $r = \frac{3x - y}{4x}$ (7)

Hence, $\frac{3x - y}{4x} = \frac{2y - x}{3y}$ (8)

Clearing of fractions, $9xy - 3y^2 = 8xy - 4x^2$ (9)

Or, $4x^2 + xy = 3y^2$ (10)

Adding $3xy$, $4x^2 + 4xy = 3xy + 3y^2$ (11)

Dividing by $x + y$, $4x = 3y$ (12)

Or, $x = \frac{3y}{4}$ (13)

Substituting in (3), $\frac{3y}{4} + y = 168$ (14)

Clearing of fractions, $3y + 4y = 672$ (15)

Uniting terms, $7y = 672$ (16)

Whence, $y = 96$ (17)

Substituting in (13), $x = \frac{3 \times 96}{4} = 72$ (18)

Substituting in (7), $r = \frac{216 - 96}{288} = \frac{5}{12}$ (19)

NOTE. If x and y be eliminated from (3), (4), and (5), we obtain the equation $12r^2 - 17r = -5$, whose roots are 1, or $\frac{5}{12}$. The former of these leads to impossible values of x and y, as it corresponds with $x + y = 0$, the factor cast out from equation (11). The factor $r - 1$, or $1 - r$, is readily discovered in the equation, and removed from it, rendering it a simple equation. Thus, after eliminating y by means of (3), (4) gives $x = \frac{112 - 168r}{1 - r}$, and (5) gives $x = \frac{42}{1 - r}$; whence, $42 = 112 - 168r$, and $r = \frac{5}{12}$.

SERIES.

ARITHMETICAL PROGRESSION.

(ART. 334, page 271.)

1. $l = a + (n-1)d = 5 + 19 \times 3 = 62$
2. $l = a + (n-1)d = 4 + 29 \times 5 = 149$
3. $l = a + (n-1)d = 10 + 4 \times (-2) = 10 - 8 = 2$
4. $l = a + (n-1)d = -6 + 14 \times \frac{1}{2} = -6 + 7 = 1$
5. $l = a + (n-1)d = 15 + 5 \times (-3) = 15 - 15 = 0$

(ART. 335, page 272.)

1. $l = a + (n-1)d = 3 + 19 \times 2 = 41$
   $S = \frac{1}{2}n(a + l) = 10 \times (3 + 41) = 10 \times 44 = 440$
2. $l = a + (n-1)d = 7 + 5 \times (-4) = 7 - 20 = -13$
   $S = \frac{1}{2}n(a + l) = 3 \times (7 - 13) = 3 \times (-6) = -18$
3. $l = a + (n-1)d = \frac{1}{2} + 19 \times 1 = 19\frac{1}{2}$
   $S = \frac{1}{2}n(a + l) = 10 \times (\frac{1}{2} + 19\frac{1}{2}) = 10 \times 20 = 200$

4. $S = \frac{1}{2}n(a + l) = 10 \times (5 + 62) = 10 \times 67 = 670$

5. $l = a + (n - 1)d = -3\frac{1}{3} + 20 \times \frac{1}{5} = -3\frac{1}{3} + 4 = \frac{2}{3}$
$S = \frac{1}{2}n(a + l) = \frac{21}{2} \times (-3\frac{1}{3} + \frac{2}{3}) = \frac{21}{2} \times (-\frac{8}{3}) = -28$

(ART. 336, page 274.)

1. $a = l - (n - 1)d = 62 - 19 \times 3 = 62 - 57 = 5$

2. $d = \frac{l - a}{n - 1} = \frac{149 - 4}{30 - 1} = \frac{145}{29} = 5$

3. $n = \frac{l - a}{d} + 1 = \frac{1 - (-6)}{\frac{1}{2}} + 1 = 14 + 1 = 15$

4. $a = \frac{2S}{n} - l = \frac{2 \times 99}{9} - 19 = 22 - 19 = 3$

5. $d = \frac{l - a}{n - 1} = \frac{2 - 10}{5 - 1} = \frac{-8}{4} = -2$

(ART. 337, page 275.)

1. $\frac{a + l}{2} = \frac{6 + 20}{2} = \frac{26}{2} = 13$

2. $d = \frac{l - a}{n - 1} = \frac{14 - 5}{4 - 1} = \frac{9}{3} = 3$
$a + d = 5 + 3 = 8;\ 8 + 3 = 11.$

3. $\frac{a + l}{2} = \frac{\frac{1}{4} + \frac{1}{9}}{2} = \frac{\frac{13}{36}}{2} = \frac{13}{72}$

4. $d = \frac{l - a}{n - 1} = \frac{3 - 1}{5 - 1} = \frac{2}{4} = \frac{1}{2}$
$a + d = 1 + \frac{1}{2} = 1\frac{1}{2};\ 1\frac{1}{2} + \frac{1}{2} = 2;\ 2 + \frac{1}{2} = 2\frac{1}{2}$

5. $\frac{\frac{a}{2} + \frac{b}{2}}{2} = \frac{a}{4} + \frac{b}{4} = \frac{1}{4}(a + b)$

6. $d = \frac{l - a}{n - 1} = \frac{\frac{11}{6} - \frac{1}{3}}{4 - 1} = \frac{\frac{9}{6}}{3} = \frac{1}{2}$
$a + d = \frac{1}{3} + \frac{1}{2} = \frac{5}{6};\ \frac{5}{6} + \frac{1}{2} = \frac{8}{6} = \frac{4}{3}$

7. $d = \frac{l-a}{n-1} = \frac{y-x}{4-1} = \frac{y-x}{3}$

$a + d = x + \frac{y-x}{3} = \frac{2x+y}{3}$; $\frac{2x+y}{3} + \frac{y-x}{3} = \frac{x+2y}{3}$

NOTE. It will be seen that the common difference is found by dividing the difference between the two given terms by the required number of means to be inserted, plus 1; that is, $d = \frac{l-a}{m+1}$, in which m represents the number of means.

PROBLEMS.

(ART. 338, page 276.)

1. $S = \frac{1}{2} n (a + l) = 6 \times (1 + 12) = 6 \times 13 = 78$

2. $l = a + (n-1) d = \frac{1}{3} + 14 \times \frac{1}{3} = \frac{1}{3} + \frac{14}{3} = \frac{15}{3} = 5$

3. $l = a + (n-1) d = 15 + 19 \times (-4) = 15 - 76 = -61$
$S = \frac{1}{2} n (a + l) = 10 \times (15 - 61) = 10 \times (-46) = -460$

4. $l = a + (n-1) d = 2 + 24 \times 3 = 2 + 72 = 74$
$S = \frac{1}{2} n (a + l) = \frac{25}{2} \times (2 + 74) = \frac{25}{2} \times 76 = 950$

5. $d = \frac{l-a}{n-1} = \frac{-\frac{1}{2} - \frac{1}{2}}{7-1} = -\frac{1}{6}$

$a + d = \frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$; $\frac{2}{6} - \frac{1}{6} = \frac{1}{6}$; $\frac{1}{6} - \frac{1}{6} = 0$; $0 - \frac{1}{6} = -\frac{1}{6}$; $-\frac{1}{6} - \frac{1}{6} = -\frac{2}{6} = -\frac{1}{3}$.

6. The values of n from the two fundamental formulas are $n = \frac{l-a}{d} + 1 = \frac{2S}{l+a}$, whence $d = \frac{l^2 - a^2}{2S - l - a}$ $= \frac{2500 - 1}{408 - 50 - 1} = \frac{2499}{357} = 7$.

Or, $n = \frac{2S}{l+a} = \frac{2 \times 204}{50 + 1} = \frac{408}{51} = 8$

$d = \frac{l-a}{n-1} = \frac{50-1}{8-1} = \frac{49}{7} = 7$

7. Art. 336, (8), $n = \frac{2S}{l+a} = \frac{2 \times 25}{7+3} = \frac{50}{10} = 5$

8. The common difference is the interest of \$20 for one year, at 4 per cent, or $20 \times \frac{4}{100} = \frac{4}{5}$.

$$l = a + (n-1)d = 20 + 19 \times \tfrac{4}{5} = 20 + \tfrac{76}{5} = 35\tfrac{1}{5}$$
$$S = \tfrac{1}{2}n(a+l) = 10 \times (20 + 35\tfrac{1}{5}) = 10 \times 55\tfrac{1}{5} = 552$$

Note. The amount of the last year's savings is simply \$20; but, as it is smaller than the amount of the first year's savings, which has 19 years' interest added to it, we use the smallest as a, or the first term, and the largest as l, or the last term.

9. Art. 336 (4), $a = \frac{2S}{n} - l = \frac{2 \times 140}{8} - 7 = 35 - 7 = 28$

Art. 336 (5), $d = \frac{l-a}{n-1} = \frac{7-28}{8-1} = \frac{-21}{7} = -3$

$a + d = 28 - 3 = 25$; $25 - 3 = 22$, etc.

10. Let $x =$ the number of seconds.

Then $8x =$ distance run by the hare.

$l = a + (n-1)d = 8 + (x-1) \times \frac{1}{2} = \frac{x+15}{2} =$ distance run by the greyhound during the last second.

$S = \frac{n}{2}(a+l) = \frac{x}{2}\left(8 + \frac{x+15}{2}\right) = \frac{x^2+31x}{4} =$ whole distance run by the greyhound.

From the conditions,	$\frac{x^2+31x}{4} = 8x + 60$
Clearing of fractions,	$x^2 + 31x = 32x + 240$
Whence,	$x^2 - x = 240$
Completing square,	$x^2 - x + \frac{1}{4} = 240\frac{1}{4} = \frac{961}{4}$
Evolving,	$x - \frac{1}{2} = \pm \frac{31}{2}$
Whence,	$x = 16$, or -15

The negative root of the equation, 15, is the answer to a problem whose conditions are the reverse of those given; that is, both must run in an opposite direction, the hare must overtake the greyhound, and the distances run by the greyhound in successive seconds must be a continuation of the given series in the other direction, $7\frac{1}{2}$ feet the first second, 7 feet the second, $6\frac{1}{2}$ the third, etc.

GEOMETRICAL PROGRESSION.

(Art. 343, p. 278.)

1. $l = ar^{n-1} = 5 \times 4^7 = 5 \times 16384 = 81920.$

2. $l = ar^{n-1} = 28672 \times (\frac{1}{4})^6 = \frac{28672}{4096} = 7.$

3. $l = ar^{n-1} = 5 \times 4^8 = 5 \times 65536 = 327680.$

4. $l = ar^{n-1} = 100 \times (\frac{2}{3})^5 = 100 \times \frac{32}{243} = \frac{3200}{243} = 13\frac{41}{243}.$

5. $l = ar^{n-1} = 3 \times 2^8 = 3 \times 256 = 768.$

(Art. 344, pp. 279, 280.)

1. $S = \frac{rl - a}{r - 1} = \frac{2 \times 1024 - 1}{2 - 1} = 2048 - 1 = 2047.$

2. $S = \frac{a(r^n - 1)}{r - 1} = \frac{6(4^8 - 1)}{4 - 1} = \frac{6 \times 65535}{3} = 131070.$

3. $S = \frac{rl - a}{r - 1} = \frac{\frac{1}{3} \times \frac{1}{162} - \frac{1}{2}}{\frac{1}{3} - 1} = \frac{-\frac{242}{486}}{-\frac{2}{3}} = \frac{121}{162}.$

4. $S = \frac{ar^n - a}{r - 1} = \frac{1 \times 3^{12} - 1}{3 - 1} = \frac{531441 - 1}{3 - 1} = \frac{531440}{2}$
$= 265720.$

5. $S = \frac{a(r^n - 1)}{r - 1} = \frac{4[(\frac{1}{2})^{16} - 1]}{\frac{1}{2} - 1} = \frac{4\left(1 - \frac{1}{2^{16}}\right)}{\frac{1}{2}} = 8\left(1 - \frac{1}{2^{16}}\right)$
$= 8 - \frac{8}{2^{16}} = 8 - \frac{2^3}{2^{16}} = 8 - \frac{1}{2^{13}} = 8 - \frac{1}{8192}.$

(Art. 345, pp. 280, 281.)

1. $S = \frac{a}{1 - r} = \frac{4}{1 - \frac{1}{2}} = \frac{4}{\frac{1}{2}} = 8.$

2. $S = \frac{a}{1 - r} = \frac{\frac{2}{7}}{1 - \frac{3}{5}} = \frac{\frac{2}{7}}{\frac{2}{5}} = \frac{5}{7}.$

3. $S = \frac{a}{1 - r} = \frac{\frac{79}{100}}{1 - \frac{1}{100}} = \frac{\frac{79}{100}}{\frac{99}{100}} = \frac{79}{99}.$

4. $S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9} = 1\frac{1}{9}$.

5. $S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{x}} = \frac{1}{1-\frac{1}{x}} \times \frac{x}{x} = \frac{x}{x-1}$.

6. $S = \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$.

(Art. **346**, page 282.)

1. $a = \frac{l}{r^{n-1}} = \frac{405}{3^4} = \frac{405}{81} = 5$.

2. $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{768}{3}\right)^{\frac{1}{8}} = (256)^{\frac{1}{8}} = 2$.

3. $r = \frac{S-a}{S-l} = \frac{9555-7168}{9555-7} = \frac{2387}{9548} = \frac{1}{4}$.

4. $a = r\,l - (r-1)\,S = 2 \times 3072 - (2-1) \times 6141$
$= 6144 - 6141 = 3$.

5. Formula (4), by a simple transposition of terms and change of signs, becomes, after dividing by r, $l = \frac{a + (r-1)\,S}{r} = \frac{2 + (3-1) \times 6560}{3} = \frac{2 + 13120}{3} = \frac{13122}{3}$
$= 4374$.

Note. Formula (4) is simply (2) cleared of fractions, etc.; hence the solution of this example is based entirely upon the second fundamental formula.

(Art. **347**, pp. 283, 284.)

1. $\sqrt{\frac{1}{3} \times \frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

2. $r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}} = \left(\frac{320}{5}\right)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = 4$;
$a\,r = 5 \times 4 = 20$; $a\,r^2 = 20 \times 4 = 80$.

3. $\sqrt{30 \times 7\frac{1}{2}} = \sqrt{225} = 15$.

4. $r = \left(\frac{l}{a}\right)^{\frac{1}{n+1}} = \left(\frac{486}{6}\right)^{\frac{1}{4}} = (81)^{\frac{1}{4}} = 3$;
$a\,r = 6 \times 3 = 18$; $a\,r^2 = 18 \times 3 = 54$; $a\,r^3 = 54 \times 3 = 162$.

PROBLEMS.

(ART. 348, page 284.)

1. $a = \frac{1}{3}$, $a\,r = 1$; $r = \frac{a\,r}{a} = \frac{1}{\frac{1}{3}} = 3$;

$a\,r^2 = 1 \times 3 = 3$; $a\,r^3 = 3 \times 3 = 9$.

2. $\frac{a\,r^4}{a\,r^3} = r^2 = \frac{300}{75} = 4$; $r = \sqrt{4} = 2$; $300 \times 2 = 600$.

3. $l = a\,r^{n-1} = 1 \times 2^9 = 512$.

4. Formula (4), Art. 344, $S = \frac{a\,(r^n - 1)}{r - 1}$, after clearing the equation of fractions and dividing by $r^n - 1$, becomes $a = \frac{(r-1)\,S}{r^n - 1} = \frac{(\frac{3}{2} - 1) \times 102.95}{(\frac{3}{2})^7 - 1} = \frac{\frac{1}{2} \times 102.95}{\frac{2187}{128} - 1}$
$= \frac{51.475}{\frac{2059}{128}} = .025 \times 128 = 3.20$.

5. Formula (5), Art. 346, $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{2048}{1}\right)^{\frac{1}{11}}$
$= (2048)^{\frac{1}{11}} = 2$.

6. Formula, Art. 345, $S = \frac{a}{1 - r} = \frac{20}{1 - \frac{19}{20}} = \frac{20}{\frac{1}{20}} = 400$.

7. By Art. 347, $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = (\frac{128}{1})^{\frac{1}{7}} = (128)^{\frac{1}{7}} = 2$;
$a = 1$; $a\,r = 1 \times 2 = 2$; $a\,r^2 = 2 \times 2 = 4$; $a r^3 = 4 \times 2 = 8$; $a r^4 = 8 \times 2 = 16$; $a r^5 = 16 \times 2 = 32$; $a r^6 = 32 \times 2 = 64$; $a\,r^7 = 64 \times 2 = 128$.

APPENDIX.

LOGARITHMS.

(368.)

2. Mantissa of 410 = .6128 $\quad$ D = 10

Correction for .4 = 4 $\quad$ .4

.6132 $\quad$ 4.0

Ans. 3.6132.

3. Mantissa of 763 = .8825 $\quad$ D = 6

Correction for .4 = 2 $\quad$ .4

.8827 $\quad$ 2.4

Ans. 3.8827.

4. Mantissa of 821 = .9143 $\quad$ D = 6

Correction for .5 = 3 $\quad$ .5

.9146 $\quad$ 3.0

Ans. 7.9146 — 10.

6. Mantissa of 106 = .0253 $\quad$ D = 41

Correction for .48 = 20 $\quad$ .48

.0273 $\quad$ 3.28

16.4

19.68

Ans. 4.0273.

8. Mantissa of 125 = .0969 $\quad$ D = 35 $\quad$ .607

Correction for .607 = 21 $\quad$ 35

.0990 $\quad$ 3.035

18.21

21.245

Ans. 0.0990.

9. Mantissa of 356 = .5514 D = 12
Correction for .7 = 8 .7
.5522 8.4

Ans. 8.5522 — 10.

10. Mantissa of 238 = .3766 D = 18
Correction for .08 = 1 .08
.3767 1.44

Ans. 8.3767 — 10.

(369.)

6.

Given logarithm = 3.0394
Next less logarithm = 3.0374; whose number = 1090
Difference of logarithms 20 = 5
Difference from column D 40 1095, Ans.

7.

Given logarithm = 2.6132
Next less logarithm = 2.6128; whose number = 410
Difference of logarithms 4 = .4
Difference from column D 10 410.4, Ans.

8.

Given logarithm = 7.9146 — 10
Next less logarithm = 7.9143 — 10; whose number = .00821
Difference of logarithms 3 = 5
Difference from column D 6 Ans. .008215

10.

Given logarithm = 0.0990
Next less logarithm = 0.0969; whose number = 1.25
Difference of logarithms 21 = 6
Difference from column D 35 1.256, Ans.

11.

$$\begin{array}{llr}
\text{Given logarithm} & = \bar{2}.5522 & \\
\text{Next less logarithm} & = \underline{\bar{2}.5514}\text{; whose number} = & .0356 \\
\text{Difference of logarithms} & \quad 8 = & \underline{\quad 7} \\
\text{Difference from column D} & \overline{12} & \text{Ans. } .03567
\end{array}$$

(370.)

2.

$$\begin{aligned}
\log \text{ of } 234 &= 2.3692 \\
\log \text{ of } 36 &= \underline{1.5563} \\
& 3.9255, \text{ whose number is } 8424, \text{ Ans.}
\end{aligned}$$

3.

$$\begin{aligned}
\log \text{ of } 59.4 &= 1.7738 \\
\log \text{ of } .000031 &= \underline{5.4914 - 10} \\
& 7.2652 - 10, \text{ whose number is } .001842, \text{ Ans.}
\end{aligned}$$

4.

$$\begin{aligned}
\log \text{ of } .224 &= 9.3502 - 10 \\
\log \text{ of } .18 &= \underline{9.2553 - 10} \\
& 9.6056 - 10, \text{ whose number is } .4033, \text{ Ans.}
\end{aligned}$$

5.

$$\begin{aligned}
\log \text{ of } 59.4 &= 1.7738 \\
\log \text{ of } .00064 &= \underline{6.8062 - 10} \\
& 8.5800 - 10, \text{ whose number is } .03802, \text{ Ans.}
\end{aligned}$$

6.

$$\begin{aligned}
\log \text{ of } 650 &= 2.8129 \\
\log \text{ of } 47 &= \underline{1.6721} \\
& 4.4850, \text{ whose number is } 30550, \text{ Ans.}
\end{aligned}$$

(371.)

2.

log of 828 = 2.9180
log of 23 = 1.3617
1.5563, whose number is 36, **Ans.**

3.

log of 589 = 2.7701
log of 31 = 1.4914
1.2787, whose number is 18.995, **Ans.**

4.

log of .00072 = 6.8573 — 10
log of .024 = 8.3802 — 10
8.4771 — 10, whose number is .03, **Ans.**

5.

log of 30550 = 4.4850
log of 47 = 1.6721
2.8129, whose number is 650, **Ans.**

7.

log of 27 = 1.4314
log of 5 = 0.6990
0.7324, whose number is 5.4, **Ans.**

8.

log of 201 = 2.3032
log of 625 = 2.7959
9.5073 — 10, whose number is .32157, **Ans.**

9.

log of 4 = 0.6021
log of 99 = 1.9956
8.6065 — 10, whose number is .04040+, **Ans.**

(372.)

2.

$$\begin{array}{r} \text{log of } 36 = 1.5563 \\ 2 \\ \hline 3.1126 \end{array}$$
3.1126, whose number is 1296, Ans.

3.

$$\begin{array}{r} \text{log of } 1.01 = 0.0043 \\ 3 \\ \hline 0.0129 \end{array}$$
0.0129, whose number is 1.0302, Ans.

4.

$$\begin{array}{r} \text{log of } .087 = 8.9395 - 10 \\ 4 \\ \hline 5.7580 - 10 \end{array}$$
5.7580 — 10, whose number is .00005728, Ans.

5.

$$\begin{array}{r} \text{log of } 3 = 0.4771 \\ 6 \\ \hline 2.8626 \end{array}$$
2.8626, whose number is 728.3, Ans.

6.

$$\begin{array}{r} \text{log of } 1.05 = .0212 \\ 20 \\ \hline .4240 \end{array}$$
.4240, whose number is 2.655+, Ans.

(373.)

2. log of 10648 = 4.0273.

$$3\,)\,\underline{4.0273}$$
1.3424, whose number is 22, Ans.

3. log of 40.96 = 1.6124.

$$2\,)\,\underline{1.6124}$$
.8062, whose number is 6.4, Ans.

4. log of $26.2 = 1.4183$.

$$3\,)\,\underline{1.4183}$$
$$.4728, \text{ whose number is } 2.97, \textbf{ Ans.}$$

6. log of $.125 = 9.0969 - 10 = 29.0969 - 30$.

$$3\,)\,\underline{29.0969 - 30}$$
$$9.6990 - 10, \text{ whose number is } .5, \textbf{ Ans.}$$

7. log of $.008649 = 7.9370 - 10 = 27.9370 - 30$.

$$3\,)\,\underline{27.9370 - 30}$$
$$9.3123 - 10, \text{ whose number is } .205+, \textbf{ Ans.}$$

8. log of $.48 = 9.6812 - 10 = 29.6812 - 30$.

$$3\,)\,\underline{29.6812 - 30}$$
$$9.8937 - 10, \text{ whose number is } .783, \textbf{ Ans.}$$

(374.)

2.

$$5^x = 100$$

Whence, $$x \log 5 = \log 100$$

Therefore, $$x = \frac{\log 100}{\log 5} = \frac{2}{.6990}$$

$$\log 2 = .3010$$
$$\log .699 = \underline{9.8445 - 10}$$

Whence, $$\log x = .4565$$

Number corresponding, $x = 2.861$, **Ans.**

3.

$$2^x = 1024$$

Whence, $$x \log 2 = \log 1024$$

Therefore, $$x = \frac{\log 1024}{\log 2} = \frac{3.0103}{.3010} = 10, \textbf{ Ans.}$$

4.

$$3^x = 20$$

Whence, $x \log 3 = \log 20$

Therefore, $x = \frac{\log 20}{\log 3} = \frac{1.3010}{.4771}$

$$\log 1.301 = .1142$$

$$\log .4771 = 9.6786 - 10$$

Whence, $\log x = .4356$

Number corresponding, $x = 2.726$, Ans.

5.

$$625^x = 3125$$

Whence, $x \log 625 = \log 3125$

Therefore, $x = \frac{\log 3125}{\log 625} = \frac{3.4948}{2.7959}$

$$\log 3.4948 = .5434$$

$$\log 2.7959 = .4465$$

Whence, $\log x = .0969$

Number corresponding, $x = 1.25$, Ans.

MULTIPLICATION OF IMAGINARY QUANTITIES.

1.

$$\sqrt{-9} \times \sqrt{-4} = (3\sqrt{-1})(2\sqrt{-1}) = 6(\sqrt{-1})^2$$
$$= 6(-1) = -6, \text{ Ans.}$$

2.

$$2\sqrt{-3} \times 3\sqrt{-2} = (2\sqrt{3}\sqrt{-1})(3\sqrt{2}\sqrt{-1})$$
$$= 6\sqrt{6}(\sqrt{-1})^2 = 6\sqrt{6}(-1)$$
$$= -6\sqrt{6}, \text{ Ans.}$$

3.

$$(1 + \sqrt{-1})(1 - \sqrt{-1}) = 1 - (\sqrt{-1})^2 \text{ (Art. 78)}$$
$$= 1 - (-1) = 1 + 1 = 2, \text{ Ans.}$$

SUPPLEMENTARY EXERCISES.

(380.)

1.

$$\begin{array}{r} 6ac - d \\ 4ac + 3d \\ 5ac - 7d \\ -4ac + 2d \\ -5ac + 4d \\ \hline 6ac + d \end{array}$$

2.

$$\begin{array}{r} 2a + 3b^2 - 5c \\ a - 2b^2 - 3c \\ \hline a + 5b^2 - 2c \end{array}$$

3.

$$\begin{array}{r} 2x^2 + 3y^m + z^3 \\ -4x^2 + 3y^m + 2z^3 - 5xy \\ -3x^2 - 2y^m + z^3 - 2xy \\ \hline -5x^2 + 4y^m + 4z^3 - 7xy \end{array}$$

4.

$$\begin{array}{r} 2x - 3y - 8z \\ 6x - 5y + 2z \\ \hline -4x + 2y - 10z \end{array}$$

5.

$$\begin{array}{l} ax + dy^2 \\ \quad - by^2 + by - dx + my \\ \hline ax - dx - by^2 + dy^2 + by + my \qquad \text{or} \\ (a - d)x - (b - d)y^2 + (b + m)y \end{array}$$

6.

$$\begin{array}{l} 3m^2n^2 - 5amn + 4a^2m^2n^2 \\ c^2m^2 \quad - 8cmn + 7m^2n^2 \\ \hline (3n^2 - c^2)m^2 + (8c - 5a)mn + (4a^2 - 7)m^2n^2 \end{array}$$

7.

$$\begin{array}{r}
x^2 + (a-b)x^2 + 5x \\
-2x^2 - 3(a-b)x^2 - 2x + 14x^3 \\
+ 7(a-b)x^2 - x - 5x^3 \\
\hline
-x^2 + 5(a-b)x^2 + 2x + 9x^3
\end{array}$$

8.

$$\begin{array}{r}
3x^4 + 5x^3 - 2x^2 \\
-5x^4 + 2x^3 - 4x^2 \\
\hline
8x^4 + 3x^3 + 2x^2
\end{array}$$

9.

$$3ax + 5bd - 2c^2x + 2cdx - bd^2$$

$$3 \,.\, 4 \,.\, 5 + 5 \,.\, 8 \,.\, 2 - 2 \,.\, 9 \,.\, 5 + 2 \,.\, 3 \,.\, 2 \,.\, 5 - 8 \,.\, 8$$

$$60 \quad + 80 \quad - 90 \quad + 60 \quad - 64$$

$$200 - 154 = 46, \text{ Ans.}$$

10.

$$\frac{4a^2 + 3b^2}{a^2 b} + \frac{3d^2(2c^2 + 3b^2 + 2x)}{a(b^2 - c^2)} - \frac{d^2 x}{b}$$

$$\frac{4 \,.\, 16 + 3 \,.\, 64}{16 \,.\, 8} + \frac{3 \,.\, \cancel{4}\,(18 + 192 + 10)}{\cancel{4}\,(64 - 9)} - \frac{8 \,.\, 5}{8}$$

$$\frac{16 \,.\, 16}{16 \,.\, 8} \quad + 3 \,.\left(\frac{220}{55}\right) \quad - 5$$

$$2 \quad + 12 \quad - 5 = 9, \text{ Ans.}$$

(381.)

1.

$$\begin{array}{r}
5ac\sqrt{x^2 - y^2} + 3bc\sqrt{x^2 + y^2} \\
3ac\sqrt{x^2 - y^2} - 4bc\sqrt{x^2 + y^2} \\
\hline
2ac\sqrt{x^2 - y^2} + 7bc\sqrt{x^2 + y^2}
\end{array}$$

2.

$$\begin{array}{r}
5a^{\frac{1}{2}}y - 3a\sqrt{y} + 8 \\
a^{\frac{1}{2}}y + 4a\sqrt{y} + 3 \\
3a^{\frac{1}{2}}y + 9a\sqrt{y} + 2 \\
\hline
9y\sqrt{a} + 10a\sqrt{y} + 13
\end{array}$$

3.

$$\text{Add}\left\{\begin{array}{l} 7a^3+9b-3cz^2 \\ 9a^3-7b+4cz^2+2 \end{array}\right\} \quad \text{Add}\left\{\begin{array}{l} -3a^2-5b+6cz^2 \\ 10a^3+7b-2cz^2+3 \end{array}\right\}$$

$$(1)\ \overline{16a^3+2b+\ cz^2+2} \quad (2)\ \overline{10a^3-3a^2+2b+4cz^2+3}$$

From (1) take (2) and we have

$$6a^3+3a^2-3cz^2-1, \text{ Ans.}$$

4.

$$\begin{array}{r} 12(ab+cd)-22(ac+bd) \\ -5(ab+cd)+77(ac+bd) \\ 19(ac+bd)+15(ad+bd) \\ \hline 7(ab+cd)+74(ac+bd)+15(ad+bd) \end{array}$$

5.

$$2a+3b-4c+(3a-2b+3c)-(2a+3b-3c)$$
$$2a+3b-4c+\ 3a-2b+3c\ -\ 2a-3b+3c$$

$$\begin{array}{r} 2a+3b-4c \\ 3a-2b+3c \\ -2a-3b+3c \\ \hline 3a-2b+2c, \end{array} \text{ Ans.}$$

6.

$$3a-[2a-2\{a-(a-1)\}+2]$$
$$=3a-\ 2a+2\{a-\ a+1\}-2$$
$$=3a-\ 2a+2a\ -2a+2\ -2$$
$$=a, \text{ Ans.}$$

7.

$$x^4-2x^3-\{6x^2+4-(2x^3-2x^2-\overline{4x-4})\}$$
$$=x^4-2x^3-\ 6x^2-4+\ 2x^3-2x^2-4x+4$$
$$=x^4-8x^2-4x, \text{ Ans.}$$

8.

$$
\begin{aligned}
& a-[5b-\{a-(3c-3b)+2c-(a-4b-c)\}] \\
&= a-5b+\{a-(3c-3b)+2c-(a-4b-c)\} \\
&= a-5b+a-3c+3b+2c-a+4b+c \\
&= a+2b, \text{ Ans.}
\end{aligned}
$$

9.

$$
\begin{aligned}
& 2a-[5b+\{3c-(a+[2b-\overline{3a+4c}])\}] \\
&= 2a-5b-\{3c-(a+[2b-\overline{3a+4c}])\} \\
&= 2a-5b-3c+(a+[2b-\overline{3a+4c}]) \\
&= 2a-5b-3c+a+2b-3a-4c \\
&= -3b-7c, \text{ Ans.}
\end{aligned}
$$

10.

$$
\begin{aligned}
& 3a-2b+4c \\
&= 3a-(2b-4c), \text{ Ans.}
\end{aligned}
$$

(382.)

1.

$$
\begin{array}{r}
5ab+4c \\
-3ac \\
\hline
-15a^2bc-12ac^2, \text{ Ans.}
\end{array}
$$

2.

$$
4a^2c^2\,\big)\,\underline{8a^2c^3-20a^3c^2m} \\
\qquad\qquad 2c-5am, \text{ Ans.}
$$

3.

$$\begin{array}{l}
3a^2b^2 - 6abc \quad + 3c^2 \\
5ab \quad - 5c \\
\hline
15a^3b^3 - 30a^2b^2c + 15abc^2 \\
\qquad\quad - 15a^2b^2c + 30abc^2 - 15c^3 \\
\hline
15a^3b^3 - 45a^2b^2c + 45abc^2 - 15c^3, \text{ Ans.}
\end{array}$$

4.

$$\begin{array}{l|l}
6a^2 + 5ab - 6b^2 & 2a + 3b \\
\cline{2-2}
6a^2 + 9ab & 3a - 2b, \text{ Ans.} \\
\cline{1-1}
\quad - 4ab - 6b^2 & \\
\quad - 4ab - 6b^2 & \\
\cline{1-1}
\end{array}$$

5.

$$\begin{array}{l}
2y^2 - 3y + 1 \\
2y^2 + 3y - 1 \\
\hline
4y^4 - 6y^3 + 2y^2 \\
\qquad + 6y^3 - 9y^2 + 3y \\
\qquad\qquad\quad - 2y^2 + 3y - 1 \\
\hline
4y^4 \qquad\qquad - 9y^2 + 6y - 1, \text{ Ans.}
\end{array}$$

6.

$$\begin{array}{l|l}
4y^4 - 9y^2 + 6y - 1 & 2y^2 + 3y - 1 \\
\cline{2-2}
4y^4 + 6y^3 - 2y^2 & 2y^2 - 3y + 1, \text{ Ans.} \\
\cline{1-1}
-6y^3 - 7y^2 + 6y & \\
-6y^3 - 9y^2 + 3y & \\
\cline{1-1}
\qquad\quad 2y^2 + 3y - 1 & \\
\qquad\quad 2y^2 + 3y - 1 & \\
\cline{1-1}
\end{array}$$

7.

$$\begin{array}{l}
a^3 - 3a^2b + 3ab^2 - b^3 \\
a^2 - 2ab + b^2 \\
\hline
a^5 - 3a^4b + 3a^3b^2 - a^2b^3 \\
\quad - 2a^4b + 6a^3b^2 - 6a^2b^3 + 2ab^4 \\
\qquad\qquad\quad a^3b^2 - 3a^2b^3 + 3ab^4 - b^5 \\
\hline
a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5, \text{ Ans.}
\end{array}$$

8.

$$
\begin{array}{l}
-6\,a^{p}\,b^{n}\,)\,12\,a^{p}\,b^{q} - 30\,a^{12}\,b^{3} \qquad + 108\,a^{\frac{1}{3}}\,b^{n} \\
\hline
\qquad -2\,b^{q-n} + 5\,a^{12-p}\,b^{3-n} - 18\,a^{\frac{1}{3}-p}, \text{ Ans.}
\end{array}
$$

9.

$$
\begin{array}{llll}
3\,a^{m-n}\,b^{-4n} & + a^{-3n}\,b^{n} & & \\
4\,a^{2m}\,b^{4n} & - 2\,a^{4n}\,b^{-3n} & & \\
\hline
12\,a^{3m-n}\,b^{0} & + 4\,a^{2m-3n}\,b^{5n} & & \\
 & - 6\,a^{m-5n}\,b^{-7n} & - 2\,a^{n}\,b^{-2n} & \\
\hline
12\,a^{3m-n} & + 4\,a^{2m-3n}\,b^{5n} & - 6\,a^{m-5n}\,b^{-7n} & - 2\,a^{n}\,b^{-2n}
\end{array}
$$

10.

$$
\begin{aligned}
& 2\,(6x - 3\,y) - \tfrac{1}{2}\,(4\,x - 4\,y) + (2\,x - 4\,y) \\
& = 12x - 6\,y \quad - 2\,x + 2\,y \quad + 2\,x - 4\,y \\
& = 12\,x - 8\,y = 4\,(3\,x - 2\,y), \text{ Ans.}
\end{aligned}
$$

(383.)

1.

$$2\,x^{-3}\,y = \frac{2\,y}{x^{3}}$$

The negative exponent indicates that a division has taken place in which the exponent of the divisor was greater than that of the dividend.

2.

$$
\begin{array}{l|l}
a^{4} + 4\,x^{4} & a^{2} - 2\,a\,x + 2\,x^{2} \\
\cline{2-2}
a^{4} - 2\,a^{3}\,x + 2\,a^{2}\,x^{2} & a^{2} + 2\,a\,x + 2\,x^{2}, \text{ Ans.} \\
\cline{1-1}
\qquad 2\,a^{3}\,x - 2\,a^{2}\,x^{2} + 4\,x^{4} & \\
\qquad 2\,a^{3}\,x - 4\,a^{2}\,x^{2} + 4\,a\,x^{3} & \\
\cline{1-1}
\qquad\qquad 2\,a^{2}\,x^{2} - 4\,a\,x^{3} + 4\,x^{4} & \\
\qquad\qquad 2\,a^{2}\,x^{2} - 4\,a\,x^{3} + 4\,x^{4} & \\
\cline{1-1}
\end{array}
$$

3.

$$\begin{array}{l}
x^{m+2}y \quad - \quad 3\,x\,y^{n-1} \\
4\,x^{m-1}y^{2} \quad - \quad 4\,x^{-1}y^{n} \\
\hline
4\,x^{2m+1}y^{3} - 12\,x^{m}\,y^{n+1} \\
\qquad\qquad\qquad\qquad - 4\,x^{m+1}y^{n+1} + 12\,x^{0}\,y^{2n-1} \\
\hline
4\,x^{2m+1}y^{3} - 12\,x^{m}\,y^{n+1} - 4\,x^{m+1}y^{n+1} + 12\,y^{2n-1}, \text{ Ans.}
\end{array}$$

4.

$$\begin{array}{l|l}
a^{2m} - 3\,a^{m}\,c^{n} + 2\,c^{2n} & a^{m} - c^{n} \\
\cline{2-2}
a^{2m} - \quad a^{m}\,c^{n} & a^{m} - 2\,c^{n}, \text{ Ans.} \\
\cline{1-1}
\qquad - 2\,a^{m}\,c^{n} + 2\,c^{2n} & \\
\qquad - 2\,a^{m}\,c^{n} + 2\,c^{2n} & \\
\cline{1-1}
\end{array}$$

5.

$$\begin{array}{l|l}
x^{n+1} + x^{n}\,y + x\,y^{n} + y^{n+1} & x^{n} + y^{n} \\
\cline{2-2}
x^{n+1} + \qquad\quad x\,y^{n} & x + y, \text{ Ans.} \\
\cline{1-1}
\qquad\quad x^{n}\,y \qquad\quad + y^{n+1} & \\
\qquad\quad x^{n}\,y \qquad\quad + y^{n+1} & \\
\cline{1-1}
\end{array}$$

6.

$$(3 - y^{n})^{2} = 9 - 6\,y^{n} + y^{2n}, \text{ Ans.}$$

$$(5\,x - 2\,y)\,(5\,x + 2\,y) = 25\,x^{2} - 4\,y^{2}, \text{ Ans.}$$

7.

$$[5 + (a + b)]\,[5 - (a + b)] = 25 - (a + b)^{2}, \text{ Ans.}$$

$$(3\,a^{m} + 2\,a^{n}\,b^{-2})\,(3\,a^{m} + 2\,a^{n}\,b^{-2}) =$$
$$9\,a^{2m} + 12\,a^{m+n}\,b^{-2} + 4\,a^{2n}\,b^{-4}, \text{ Ans.}$$

8.

$(2x + b)(2x - b)(4x^2 - b^2) = ?$

$(2x + b)(2x - b) = 4x^2 - b^2$ by Theorem III.: then $(4x^2 - b^2)(4x^2 - b^2) = 16x^4 - 8x^2b^2 + b^4$, Ans.

9.

$$9a^{-2m} + 12a^{-m}b^{\frac{1}{2}} + 4b, \text{ Ans.}$$

10.

$$\begin{aligned}&(a + b)(b + c) - (c + d)(d + a) - (a + c)(b - d) \\ = &(ab + ac + b^2 + bc) - (cd + ca + d^2 + da) \\ &\qquad - (ab - ad + bc - cd)\end{aligned}$$

Free from parenthesis and collect terms;

$$\begin{aligned}ab - ac + b^2 + bc - cd - ca - d^2 - da - ab + ad \\ - bc + cd \\ = b^2 - 2ca - d^2, \text{ Ans.}\end{aligned}$$

11.

$x(x + 1)(x + 2)(x + 3) + 1 = ?$

$$\begin{aligned}x(x + 1) &= x^2 + x \\ (x + 2)(x + 3) &= x^2 + 5x + 6 \\ (x^2 + x)(x^2 + 5x + 6) + 1 &= x^4 + 5x^3 + 6x^2 + x^3 \\ &\quad + 5x^2 + 6x + 1 \\ &= x^4 + 6x^3 + 11x^2 + 6x + 1, \text{ Ans.}\end{aligned}$$

12.

Removing the { } changes the last ( ) to —.

$$\begin{array}{rllllll} (a + b + c)^2 = & a^2 + b^2 + c^2 & + 2ab & + 2ac & + 2bc \\ -a(b + c - a) = & a^2 & - ab & - ac & \\ -b(a + c - b) = & \quad + b^2 & - ab & & - bc \\ -c(a + b - c) = & \qquad + c^2 & & - ac & - bc \\ \hline \text{Adding, we have} & \multicolumn{4}{l}{2a^2 + 2b^2 + 2c^2, \text{ Ans.}} \end{array}$$

13.

$$
\begin{array}{l}
ab + bc + ca \\
a \quad + b \quad + c \\
\hline
a^2b + abc + a^2c + ab^2 + b^2c + bca + abc + bc^2 + c^2a \\
\quad - abc
\end{array}
$$

$$
\begin{array}{l|l}
a^2b + a^2c + ab^2 + 2abc + bc^2 + c^2a + b^2c & a + b \\
\cline{2-2}
a^2b \qquad + ab^2 & ab + ac \\
\cline{1-1}
\qquad a^2c \qquad + 2abc & \quad + bc \\
\qquad a^2c \qquad + \ abc & \quad + \ c^2 \\
\cline{1-1}
\qquad\qquad abc + b^2c & \\
\qquad\qquad abc + b^2c & \\
\cline{1-1}
\qquad\qquad\qquad c^2a + c^2b & \\
\qquad\qquad\qquad c^2a + c^2b & \\
\cline{1-1}
\end{array}
$$

14.

$$
\begin{array}{l}
(ax + by)^2 = a^2x^2 + 2abxy + b^2y^2 \\
(ay - bx)^2 = a^2y^2 - 2abxy + b^2x^2 \\
\qquad\qquad\quad c^2y^2 \qquad\qquad + c^2x^2 \\
\hline
\{(a^2 + b^2 + c^2)\,x^2 + (a^2 + b^2 + c^2)\,y^2\} \div (x^2 + y^2) \\
\qquad = a^2 + b^2 + c^2, \text{ Ans.}
\end{array}
$$

15.

$$[(x^3 - 12x + 16)\,(x^3 - 12x - 16)] \div (x^2 - 16) = ?$$

$$
\begin{array}{l}
x^3 - 12x + 16 \\
x^3 - 12x - 16 \\
\hline
x^6 - 12x^4 + 16x^3 \\
\quad - 12x^4 + 144x^2 - 192x \\
\qquad\quad - 16x^3 + 192x - 256
\end{array}
$$

$$
\begin{array}{l|l}
x^6 - 24x^4 + 144x^2 \qquad\qquad - 256 & x^2 - 16 \\
\cline{2-2}
x^6 - 16x^4 & x^4 - 8x^2 + 16, \text{ Ans.} \\
\cline{1-1}
\quad - 8x^4 + 144x^2 & \\
\quad - 8x^4 + 128x^2 & \\
\cline{1-1}
\qquad\qquad 16x^2 \qquad\qquad - 256 & \\
\qquad\qquad 16x^2 \qquad\qquad - 256 & \\
\cline{1-1}
\end{array}
$$

(384.)

1.

$$15\,x^{2n}\,y^{3m}\,z^{2} = 3\,.\,5\,.\,x^{n}\,.\,x^{n}\,.\,y^{m}\,.\,y^{m}\,.\,y^{m}\,.\,z\,.\,z, \text{ Ans.}$$

2.

$$13\,a^{2}y - 39\,a^{3}x = 13\,a\,.\,a\,.\,(y - 3\,a\,x)$$

Take out $13\,a^{2}$; the other factor $= (y - 3\,a\,x)$, Ans.

3.

$$4\,a^{2}x^{2} + 6\,a^{3}x^{4} - 2\,a\,x\,y$$
$$= 2\,a\,x\,(2\,a\,x + 3\,a^{2}x^{3} - y), \text{ Ans.}$$

4.

$$36\,a^{2m}\,b^{2}\,x^{\frac{1}{2}} - 42\,a^{3m}\,b^{3}\,x^{\frac{1}{3}} + 48\,a^{2m}\,b^{2}\,x$$
$$= 6\,a^{2m}\,b^{2}\,x^{\frac{1}{3}}\,(6\,x^{\frac{1}{6}} - 7\,a^{m}\,b + 8\,x^{\frac{2}{3}}), \text{ Ans.}$$

5.

$$60\,a^{3}\,x^{\frac{3}{5}}\,y^{-3m} = ?$$
$$60 = 5\,.\,2\,.\,2\,.\,3$$
$$a^{3} = a\,.\,a\,.\,a$$
$$x^{\frac{3}{5}} = \sqrt[5]{x\,.\,x\,.\,x}$$
$$y^{-3m} = \frac{1}{y^{m}\,.\,y^{m}\,.\,y^{m}}$$

$$\text{Ans. } \frac{5\,.\,2\,.\,2\,.\,3\,.\,a\,.\,a\,.\,a\,.\,x^{\frac{1}{5}}\,.\,x^{\frac{1}{5}}\,.\,x^{\frac{1}{5}}}{y^{m}\,.\,y^{m}\,.\,y^{m}}.$$

6.

$$(18\,a^{2}\,c^{2})\left(\frac{4\,a^{2}\,c}{b^{3}} + 7\,a\,d + 9\,c^{2}\right), \text{ Ans.}$$

7.

$$\begin{array}{l|l} 42\,a\,b\,x^{3m} + 56\,a^{\frac{3}{2}}\,b^{-3}\,x^{4m} & 7\,a^{\frac{1}{2}}\,b^{-2}\,x^{2m} \\ \hline \underline{42\,a\,b\,x^{3m} + 56\,a^{\frac{3}{2}}\,b^{-3}\,x^{4m}} & 6\,a^{\frac{1}{2}}\,b^{3}\,x^{m} + 8\,a^{\frac{1}{2}}\,b^{-1}\,x^{2m}, \text{ Ans.} \end{array}$$

8.

$$16\,y = 4\,y^{\frac{1}{2}} \,.\, 4\,y^{\frac{1}{2}}, \text{ Ans.}$$
$$\text{It also} = 2\,y^{\frac{1}{4}} \,.\, 2\,y^{\frac{1}{4}} \,.\, 2\,y^{\frac{1}{4}} \,.\, 2\,y^{\frac{1}{4}}, \text{ Ans.}$$

9.

$$a\,x^2 - b\,x^2 + c\,x^3$$
$$= x^2\,(a - b + c\,x), \text{ Ans.}$$

10.

$$5\,a^5\,x^3 + 10\,a^4\,x^4 + 5\,a^3\,x^5$$
$$= 5\,a^3\,x^3\,(a^2 + 2\,a\,x + x^2)$$
$$= 5\,a^3\,x^3\,(a + x)^2$$
$$= 5 \,.\, a^3\,x^3 \,.\, (a + x)\,(a + x), \text{ Ans.}$$

(385.)

1.

$$4\,x^2 - 4\,x\,y + y^2 = (2\,x - y)\,(2\,x - y), \text{ Ans.}$$

2.

$$1 - a^4 = (1 + a^2)\,(1 + a)\,(1 - a), \text{ Ans.}$$
$$1 - y^6 = (1 + y)\,(1 - y + y^2 - y^3 + y^4 - y^5), \text{ Ans.}$$

3.

$$16\,x^2 - 40\,x\,y + 25\,y^2 = (4\,x - 5\,y)\,(4\,x - 5\,y), \text{ Ans.}$$

4.

$$7\,a^4\,b^3\,x^4 - 14\,a^3\,b^3\,x^5 + 7\,a^2\,b^3\,x^6 =$$
$$7\,a^2\,b^3\,x^4\,(a^2 - 2\,a\,x + x^2) =$$
$$(7\,a^2\,b^3\,x^4)\,(a - x)\,(a - x), \text{ Ans.}$$

5.

$$x^2\,y^2 + 32\,x\,y + 256 = (x\,y + 16)^2, \text{ Ans.}$$

6.

$$4x^4 - 60mnx^2 + 225m^2n^2 =$$
$$(2x^2 - 15mn)(2x^2 - 15mn), \text{ Ans.}$$

7.

$$x^7 + y^7 = (x + y)(x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6)$$
$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^5 + y^4), \text{ Ans.}$$

8.

$$\frac{x^8 - z^8}{x^2 - z^2} = \frac{(x^4 + z^4)(x^2 + z^2)(x^2 - z^2)}{x^2 - z^2}$$
$$= (x^4 + z^4)(x^2 + z^2), \text{ Ans.}$$

9.

$$4x^2 - 225y^4 = (2x + 15y^2)(2x - 15y^2), \text{ Ans.}$$
$$1 - 196x^2y^4 = (1 + 14xy^2)(1 - 14xy^2), \text{ Ans.}$$

10.

$$m^2 - (x - y)^2 = [m + (x - y)][m - (x - y)]$$
$$= (m + x - y)(m - x + y), \text{ Ans.}$$
$$x^{\frac{2}{3}} - y^{\frac{2}{3}} = (x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}}), \text{ Ans.}$$

(386.)

1.

$$bc + bd + cx + dx = b(c + d) + x(c + d)$$
$$= (b + x)(c + d), \text{ Ans.}$$

2.

$$x^3 + x^2y + xy^2 + y^3 = x^2(x + y) + y^2(x + y)$$
$$= (x^2 + y^2)(x + y), \text{ Ans.}$$

3.

$$x^2 + 2x - xy - 2y = x(x + 2) - y(x + 2)$$
$$= (x - y)(x + 2), \text{ Ans.}$$

4.

$$a^3 - a^2 b + a b^2 - b^3 = a^2 (a - b) + b^2 (a - b)$$
$$= (a^2 + b^2)(a - b), \text{ Ans.}$$

5.

$$6n - 21 m^2 n - 8m + 28 m^3 =$$
$$3n (2 - 7m^2) - 4m (2 - 7m^2) =$$
$$(3n - 4m)(2 - 7m^2), \text{ Ans.}$$

6.

$$x^2 + 2xy + y^2 - 4 = (x + y)^2 - 4$$
$$= (x + y + 2)(x + y - 2), \text{ Ans.}$$

7.

$$a^2 - 2ab + b^2 - 25 = (a - b)^2 - 25$$
$$= (a - b + 5)(a - b - 5), \text{ Ans.}$$

8.

$$9c^2 + 6cd + d^2 - 1 = (3c + d)^2 - 1$$
$$= (3c + d + 1)(3c + d - 1), \text{ Ans.}$$

9.

$$x - 2x^{\frac{1}{2}} y^2 + y^4 = (x^{\frac{1}{2}} - y^2)(x^{\frac{1}{2}} - y^2), \text{ Ans.}$$

10.

$$25x^2 + 70xyz + 49y^2 z^2 =$$
$$(5x + 7yz)(5x + 7yz), \text{ Ans.}$$

(387.)

1. $x^2 + 5x + 6 = (x + 2)(x + 3)$, Ans.
2. $x^2 + 23x + 102 = (x + 6)(x + 17)$ Ans.
3. $x^2 + 13x + 36 = (x + 9)(x + 4)$, Ans.
4. $c^2 - 18c + 32 = (c - 16)(c - 2)$, Ans.

5. $x^2 + x - 42 = (x + 7)(x - 6)$, Ans.

6. $y^2 - 9y - 90 = (y - 15)(y + 6)$, Ans.

7. $x^2 y^6 + 2xy^3 - 120 = (xy^3 + 12)(xy^3 - 10)$, Ans.

8. $x^2 - 29x + 120 = (x - 24)(x - 5)$, Ans.

9. $x^2 + 25nx + 100n^2 = (x + 20n)(x + 5n)$, Ans.

10. $a^2 b^4 - 7ab^2 - 144 = (ab^2 - 16)(ab^2 + 9)$, Ans.

11. $x^2 y^2 + 12xy + 27 = (xy + 9)(xy + 3)$, Ans.

12. $x^2 - 2xy^2z - 48y^4z^2 = (x - 8y^2z)(x + 6y^2z)$, Ans.

13. $y^2 - y - 210 = (y - 15)(y + 14)$, Ans.

14. $x^2y^2 - 24xyz + 143z^2 = (xy - 13z)(xy - 11z)$, Ans.

15. $a^2 + 13a - 48 = (a + 16)(a - 3)$, Ans.

(388.)

1.

$$42ab^2c^3 = 7 \,.\, 3 \,.\, 2a \,.\, b^2 \,.\, c^3$$
$$14ab^2 - 21b^3c^2 = 7b^2(2a - 3bc^2)$$
$$7b^2, \text{ Ans.}$$

2.

$$\frac{40a^3b^3}{8a^3b^2} = 5b. \qquad \frac{40a^3b^3}{20a^2b^3} = 2a.$$

3.

$$3x^3y - 6xy^3 = 3xy(x^2 - 2y^2)$$
$$3xy - 9x^2y^2 = 3xy(1 - 3xy)$$
$$\text{G. C. D.} = 3xy, \text{ Ans.}$$

4.

$$\left.\begin{array}{l} xy^2 + xyc = xy(y + c) \\ xy^2 - xyc = xy(y - c) \\ xy^3 - xyc^2 = xy(y^2 - c^2) \end{array}\right\} \text{L. C. M.} = xy(y^2 - c^2), \text{ Ans.}$$

5.

$$\begin{array}{rl}
12\,a^2 b c x^2 = & 2 \,.\, 2 \,.\, 3 \,. \qquad a^2 \,.\, b \,.\, c \,.\, x^2 \\
20\,b^2 c^2 x^2 = & 2 \,.\, 2 \,.\, 5 \,. \qquad\quad b^2 \,.\, c^2 \,.\, x^2 \\
32\,a^5 b^2 c^3 x = & \underline{2 \,.\, 2 \,.\, 2 \,.\, 2 \,.\, 2 \,.\, a^5 \,.\, b^2 \,.\, c^3 \,.\, x} \\
\text{G. C. D.} = & 2 \,.\, 2 \,.\, b \,.\, c \,.\, x = 4\,b c x, \text{ Ans.}
\end{array}$$

6.

$$\begin{array}{rl}
x^2 + 5x + 4 = & (x+4)\ (x+1) \\
x^2 + 2x - 8 = & (x+4)\ (x-2) \\
x^2 + 7x + 12 = & (x+4)\ (x+3) \\
\hline
\text{L. C. M.} = & (x+1)\ (x-2)\ (x+3)\ (x+4), \text{ Ans.}
\end{array}$$

7.

$$\begin{array}{rl}
a^2 x^2 + 5\,a^2 x - 24\,a^2 = & a^2\,(x^2 + 5\,x - 24) \\
= & a^2\,(x+8)\ (x-3) \\
ab^2 x^2 + 23\,a b^2 x + 120\,a b^2 = & a\,b^2\,(x^2 + 23\,x + 120) \\
= & a\,b^2\,(x+15)\ (x+8)
\end{array}$$

$$\text{G. C. D.} = a\,(x+8), \text{ Ans.}$$

8.

$$\begin{array}{rl}
3\,(a+b)^2 = & 3 \,. \qquad (a+b)\ (a+b) \\
6\,(a^2-b^2) = & 3 \,.\, 2 \,. \quad (a+b)\ (a-b) \\
12\,(a-b)^2 = & \underline{2 \,.\, 2 \,.\, 3\ (a-b)\ (a-b)} \\
\text{L. C. M.} = & 2 \,.\, 2 \,.\, 3 \,.\, (a+b)^2\,(a-b)^2 = \\
& 12\,(a^2-b^2)\ (a^2-b^2) = 12\,(a^2-b^2)^2, \text{ Ans.}
\end{array}$$

9.

$$\begin{array}{rl}
a x + b x + a y + b y = & x\,(a+b) + y\,(a+b) \\
a c + b c + a d + b d = & c\,(a+b) + d\,(a+b)
\end{array}$$

$$\text{G. C. D.} = a + b, \text{ Ans.}$$

10.

$$x + 1 = x + 1$$
$$x^2 - 1 = (x + 1)\ (x - 1)$$
$$x^3 - 1 = (x - 1)\ (x^2 + x + 1)$$
$$\text{L. C. M.} = (x^2 + x + 1)\ (x - 1)\ (x + 1) =$$
$$(x^3 - 1)\ (x + 1), \text{ Ans.}$$

(389.)

1.

$$\begin{array}{r|l}
x^4 - x^3 + 2x^2 + x + 3\ ) & x^4 - 2x^3 - \quad x - 2\ (1 \\
 & \underline{x^4 - \ x^3 + 2x^2 + \ x + 3} \\
 & \quad - \ x^3 - 2x^2 - 2x - 5
\end{array}$$

$$\begin{array}{r|l}
x^3 + 2x^2 + 2x + 5\ ) & x^4 - \ x^3 + 2x^2 + \ x + 3\ (x \\
 & \underline{x^4 + 2x^3 + 2x^2 + 5x} \\
 & -3x^3 \qquad\quad - 4x + \ 3\ (-3 \\
 & \underline{-3x^3 - 6x^2 - 6x - 15} \\
 & \qquad\qquad 6x^2 + 2x + 18 = \\
 & \qquad\qquad\qquad 2\ (3x^2 + x + 9)
\end{array}$$

$$\begin{array}{r|l}
 & x^3 + 2x^2 + 2x + \ 5 \\
 & \underline{\qquad\qquad\qquad\qquad 3} \\
3x^2 + x + 9\ ) & 3x^3 + 6x^2 + 6x + 15\ (x + 5 \\
 & \underline{3x^3 + \ x^2 + 9x} \\
 & \qquad 5x^2 - 3x + 15 \\
 & \underline{\qquad\qquad\qquad 3} \\
 & \quad 15x^2 - 9x + 45 \\
 & \quad \underline{15x^2 + 5x + 45} \\
 & \qquad\quad - 14x
\end{array}$$

The polynomials are prime to each other.

G. C. D. $= 1$, Ans.

2.

$$
\begin{array}{r}
x^3 + 3x^2 + 4x + 12\ )\ x^3 + 4x^2 + 4x + 3\ (1 \\
\underline{x^3 + 3x^2 + 4x + 12} \\
x^2 \qquad - 9
\end{array}
$$

$$
\begin{array}{r}
x^2 - 9\ )\ x^3 + 3x^2 + 4x + 12\ (x + 3 \\
\underline{x^3 \qquad - 9x \qquad} \\
3x^2 + 13x + 12 \\
\underline{3x^2 \qquad - 27} \\
13x + 39
\end{array}
$$

$$
\begin{array}{r}
x + 3\ )\ x^2 - 9\ \ (x - 3 \\
\underline{x^2 + 3x} \\
-3x - 9 \\
\underline{-3x - 9}
\end{array}
$$

G. C. D. $= x + 3$, Ans.

3.

Take $2ax^2$ out of first.

" ax " second.

Then find G. C. D. of $x^2 + 7x + 12$ and $x^2 - 2x - 15$, which is $x + 3$.

As ax is common, restore it, and

G. C. D. $= ax(x + 3)$, Ans.

4.

$acy^2 + ady^2 + bcy^2 + bdy^2$

Take out y^2,

$ac + ad + bc + bd$

Factoring,

$a(c + d) + b(c + d) =$

$(a + b)(c + d)$

$amx - anx + bmx - bnx$

Take out x.

$am - an + bm - bn$

Factoring,

$a(m - n) + b(m - n) =$

$(a + b)(m - n)$

G. C. D. $= a + b$, Ans.

5.

$$
\begin{array}{l}
x^4 - x^3 - 5x^2 + 2x + 6 \,)\, x^4 + x^3 - x^2 - 2x - 2 \,(\, 1 \\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{x^4 - x^3 - 5x^2 + 2x + 6} \\
\qquad\qquad\qquad\qquad\qquad\qquad 2x^3 + 4x^2 - 4x - 8
\end{array}
$$

$$
\begin{array}{l}
x^3 + 2x^2 - 2x - 4 \,)\, x^4 - x^3 - 5x^2 + 2x + 6 \,(\, x - 3 \\
\qquad\qquad\qquad\qquad \underline{x^4 + 2x^3 - 2x^2 - 4x} \\
\qquad\qquad\qquad\qquad\quad - 3x^3 - 3x^2 + 6x + 6 \\
\qquad\qquad\qquad\qquad\quad \underline{- 3x^3 - 6x^2 + 6x + 12} \\
\qquad\qquad\qquad\qquad\qquad\qquad 3x^2 \qquad - 6
\end{array}
$$

$$
\begin{array}{l}
x^2 - 2 \,)\, x^3 + 2x^2 - 2x - 4 \,(\, x + 2 \\
\qquad\quad \underline{x^3 \qquad - 2x} \\
\qquad\qquad\;\; 2x^2 \qquad - 4 \\
\qquad\qquad\;\; \underline{2x^2 \qquad - 4}
\end{array}
$$

G. C. D. $= x^2 - 2$, Ans.

6.

NOTE. — If factors cannot be readily found by inspection, find the G. C. D. and then use it.

Factoring,

$$6x^2 - x - 1 = (3x + 1)(2x - 1)$$

$$2x^2 + 3x - 2 = (x + 2)(2x - 1)$$

L. C. M. $= (3x + 1)(2x - 1)(x + 2)$, Ans.

7.

Factoring,

$$4x^2 - 4x + 1 = (2x - 1)^2$$

$$4x^2 - 1 = (2x - 1)(2x + 1)$$

$$4x^2 + 4x + 1 = (2x + 1)^2$$

$$
\begin{aligned}
\text{L. C. M.} &= (2x - 1)(2x - 1)(2x + 1)(2x + 1) \\
&= (4x^2 - 1)(4x^2 - 1) \\
&= 16x^4 - 8x^2 + 1, \text{ Ans.}
\end{aligned}
$$

8.

$$ax - ay - bx + by = a(x - y) - b(x - y)$$
$$x^2 - 2xy + y^2 = (x - y)(x - y)$$
$$\text{L. C. M.} = (a - b)(x - y)^2, \text{ Ans.}$$

9.

$$6x^2 + 13x - 28 = (3x - 4)(2x + 7)$$
$$12x^2 - 31x + 20 = (3x - 4)(4x - 5)$$
$$\text{L. C. M.} = (3x - 4)(2x + 7)(4x - 5), \text{ Ans.}$$

10.

$$8x^2 + 30x + 7 = (2x + 7)(4x + 1).$$
$$12x^2 - 29x - 8 = (3x - 8)(4x + 1).$$
$$\text{L. C. M.} = (2x + 7)(4x + 1)(3x - 8), \text{ Ans.}$$

(390.)

1.

$$\frac{35\,m x^2 y^2}{105\,m^2 x y^5} = \frac{x}{3my^3}, \text{ Ans.}$$

2.

$$\frac{2xz - y^2}{2z} = x - \frac{y^2}{2z}, \text{ Ans.}$$

3.

$$\frac{ax^3m - am}{x - 1} = \frac{am(x^3 - 1)}{x - 1} = am(x^2 + x + 1), \text{ Ans.}$$

4.

$$\frac{6a^2x - 5ax^2 + 4x^3}{}$$

$$\begin{array}{r} 2a - x\,)\,6a^2x - 5ax^2 + 4x^3\,\left(3ax - x^2 + \dfrac{3x^3}{2a - x}\right. \\ \underline{6a^2x - 3ax^2} \qquad\qquad\qquad\qquad\qquad \\ -2ax^2 + 4x^3 \qquad\qquad\qquad\qquad\qquad \\ \underline{-2ax^2 + x^3} \qquad\qquad\qquad\qquad\qquad \\ 3x^3 \qquad\qquad\qquad\qquad\qquad \end{array}$$

5.

$$\frac{ax+b}{c^2-d^2},\ \frac{a}{c-d},\ \frac{b}{c+d} = \frac{ax+b}{c^2-d^2},\ \frac{a(c+d)}{c^2-d^2},\ \frac{b(c-d)}{c^2-d^2},\ \text{Ans.}$$

6.

$$\frac{m^2 - 10m + 16}{m^2 + m - 72} = \frac{(m-2)(m-8)}{(m+9)(m-8)} = \frac{m-2}{m+9},\ \text{Ans.}$$

7.

$$\frac{ac - bc - ad + bd}{ac + ad - bc - bd} = \frac{c(a-b) - d(a-b)}{c(a-b) + d(a-b)} = \frac{c-d}{c+d},\ \text{Ans.}$$

8.

$$\frac{4c^2 - 20c + 25}{25 - 4c^2} = \frac{(2c-5)(2c-5)}{(5-2c)(5+2c)}$$

$$= \frac{(5-2c)(5-2c)}{(5-2c)(5+2c)}$$

$$= \frac{5-2c}{5+2c},\ \text{Ans.}$$

9.

$$x^3 - 4x^2 + 2x + 3\,)\,x^3 + x^2 - 3x - 2\,(\,1$$
$$\underline{x^3 - 4x^2 + 2x + 3}$$
$$5x^2 - 5x - 5$$

$$x^2 - x - 1\,)\,x^3 - 4x^2 + 2x + 3\,(\,x - 3$$
$$\underline{x^3 - x^2 - x}$$
$$-3x^2 + 3x + 3$$
$$\underline{-3x^2 + 3x + 3}$$

$$\therefore \text{G. C. D.} = x^2 - x - 1$$

$$x^2 - x - 1\,)\,x^3 + x^2 - 3x - 2\,(\,x + 2$$
$$\underline{x^3 - x^2 - x}$$
$$2x^2 - 2x - 2$$
$$\underline{2x^2 - 2x - 2}$$

$$\therefore \text{The fraction} = \frac{x+2}{x-3}, \text{ Ans.}$$

10.

$$3x - 2 - \frac{3}{2x-1} =$$
$$\frac{6x^2 - 4x - 3x + 2 - 3}{2x-1} =$$
$$\frac{6x^2 - 7x - 1}{2x-1}, \text{ Ans.}$$

$$a - b - \frac{a^2+b^2}{a+b} = ?$$
$$(a-b)\,(a+b) = a^2 - b^2$$
$$\frac{a^2 - b^2 - a^2 - b^2}{a+b} = \frac{-2b^2}{a+b}, \text{ Ans.}$$

11.

$$\frac{x+3}{x^2 - 3x + 2}, \quad \frac{x+1}{x^2 - 5x + 6} =$$
$$\frac{x+3}{(x-2)\,(x-1)}, \quad \frac{x+1}{(x-3)\,(x-2)} =$$
$$\frac{(x+3)\,(x-3)}{(x-1)\,(x-2)\,(x-3)}, \quad \frac{(x+1)\,(x-1)}{(x-1)\,(x-2)\,(x-3)}, \text{ Ans.}$$

12.

$$\frac{2a}{a^2+a-6}, \frac{4c}{a^2-4} = \frac{2a}{(a+3)(a-2)}, \frac{4c}{(a-2)(a+2)} =$$

$$\frac{2a(a+2)}{(a+3)(a-2)(a+2)}, \frac{4c(a+3)}{(a+3)(a-2)(a+2)}, \text{ Ans.}$$

(391.)

1.

$$\frac{2a}{a+x} + \frac{3x}{a-x} + \frac{3x^2+a^2}{a^2-x^2}$$

$$= \frac{2a(a-x) + 3x(a+x) + 3x^2 + a^2}{a^2-x^2}$$

$$= \frac{2a^2 - 2ax + 3ax + 3x^2 + 3x^2 + a^2}{a^2-x^2}$$

$$= \frac{3a^2 + ax + 6x^2}{a^2-x^2}, \text{ Ans.}$$

2.

$$\frac{7}{5a+4b} - \frac{7}{5a-4b} = \frac{7(5a-4b) - 7(5a+4b)}{25a^2 - 16b^2}$$

$$= \frac{-56b}{25a^2 - 16b^2}, \text{ Ans.}$$

3.

$$\frac{a^2+2ax+x^2}{a-x} - \frac{a^2-2ax+x^2}{a+x}$$

$$= \frac{(a+x)^2(a+x) - (a-x)^2(a-x)}{a^2-x^2}$$

$$= \frac{(a^3+3a^2x+3ax^2+x^3) - (a^3-3a^2x+3ax^2-x^3)}{a^2-x^2}$$

$$= \frac{6a^2x+2x^3}{a^2-x^2}, \text{ Ans.}$$

4.

$$\left(5a - \frac{x-7}{4}\right) - \left(2a + \frac{x-3}{12}\right) = 3a - \frac{x-7}{4} - \frac{x-3}{12}$$

$$= \frac{36a - 3x + 21 - x + 3}{12} = \frac{36a - 4x + 24}{12}, \text{ Ans.}$$

5.

$$\frac{3+2y}{2-y} + \frac{16y - y^2}{y^2-4} - \frac{2-3y}{2+y}$$

$$= \frac{(3+2y)(-y-2) + (16y - y^2) - (2-3y)(y-2)}{y^2-4}$$

$$= \frac{-3y - 6 - 2y^2 - 4y + 16y - y^2 - 2y + 4 + 3y^2 - 6y}{y^2-4}$$

$$= \frac{y-2}{y^2-4} = \frac{1}{y+2}, \text{ Ans.}$$

6.

$$\frac{x}{1+x} - \frac{x}{1-x} + \frac{x^2}{x^2-1} = \frac{x(1-x) - x(1+x) - x^2}{1-x^2}$$

$$= \frac{x - x^2 - x - x^2 - x^2}{1-x^2} = \frac{-3x^2}{1-x^2}, \text{ Ans.}$$

7.

$$\frac{2x-6}{x^2+3x+2} - \frac{x+2}{x^2-2x-3} - \frac{x+1}{x^2-x-6}$$

$$= \frac{2x-6}{(x+2)(x+1)} - \frac{x+2}{(x-3)(x+1)} - \frac{x+1}{(x-3)(x+2)}$$

$$= \frac{2(x-3)^2 - (x+2)(x+2) - (x+1)^2}{(x+1)(x+2)(x-3)}$$

$$= \frac{2x^2 - 12x + 18 - x^2 - 4x - 4 - x^2 - 2x - 1}{(x+1)(x+2)(x-3)}$$

$$= \frac{-18x + 13}{(x+1)(x+2)(x-3)}, \text{ Ans.}$$

8.

$$\frac{5a+1}{3a+3}+\frac{3a-1}{2-2a}=\frac{5a+1}{3(a+1)}-\frac{3a-1}{2(a-1)}$$

$$=\frac{2(5a+1)(a-1)-3(a+1)(3a-1)}{6(a^2-1)}$$

$$=\frac{10a^2-8a-2-9a^2-6a+3}{6(a^2-1)}=\frac{a^2-14a+1}{6(a^2-1)},\ \text{Ans.}$$

9.

$$\frac{3a+1}{12a}-\frac{2b-1}{8b}+\frac{4c-1}{16c}-\frac{6d+1}{24d}$$

$$=\frac{4bcd(3a+1)-6acd(2b-1)+3abd(4c-1)-2abc(6d+1)}{48abcd}$$

$$=\frac{12abcd+4bcd-12abcd+6acd+12abcd-8abd-12abcd-2abc}{48abcd}$$

$$=\frac{4bcd+6acd-3abd-2abc}{48abcd},\ \text{Ans.}$$

10.

$$\frac{1}{(a-b)(b-c)}+\frac{1}{(b-a)(a-c)}-\frac{1}{(c-a)(c-b)}$$

$$=\frac{a-c+c-b-a+b}{(a-b)(a-c)(b-c)}=0,\ \text{Ans.}$$

(392.)

1.

$$\left(\frac{a^2-x^2}{a}\right)\frac{(a^2+x^2)}{ax}\,\frac{a^2x^2}{y^2}=\frac{x(a^4-x^4)}{y^2},\ \text{Ans.}$$

2.

$$\frac{4x}{9}\div\frac{4x^2}{45}=\frac{5}{x};\quad \frac{2x}{21a^2}\div\frac{2}{ax^3}=\frac{ax^3}{21a^2}=\frac{x^3}{21a},\ \text{Ans.}$$

3.

$$\frac{3x^3}{a^3+x^3} \times \frac{a+x}{x} = \frac{3x}{a^2-ax+x^2}; \quad \frac{a+x}{c+d} \times \frac{c^2-d^2}{(a+x)^2} = \frac{c-d}{a+x}.$$

4.

$$\left(3a+\frac{4x^2}{5c}\right)\left(3a-\frac{4x^2}{5c}\right) = 9a^2-\frac{16x^4}{25c^2}, \text{ Ans.}$$

5.

$$\frac{n+\frac{1}{n}}{\frac{a}{n}+\frac{1}{n}} = \frac{\frac{n^2+1}{n}}{\frac{a+1}{n}} = \frac{n^2+1}{a+1}, \text{ Ans.}$$

6.

$$\frac{a^{-2}b^{-3}}{a^{-2}e^{-1}} = \frac{a^2e}{a^2b^3} = \frac{e}{b^3}, \text{ Ans.}$$

$$\frac{x^2+y^{-3}}{mn^{-4}e^{-2}} = \frac{\left(x^2+\frac{1}{y^3}\right)n^4e^2}{m} = \frac{n^4e^2(x^2y^3+1)}{my^3}, \text{ Ans.}$$

7.

$$\left(\frac{x^2-3x+2}{x^2-8x+15}\right)\frac{(x^2-7x+12)}{(x^2-5x+4)}\left(\frac{x^3-5x^2}{x^2-4}\right) =$$

$$\frac{(x-2)(x-1)(x-4)(x-3)x^2(x-5)}{(x-5)(x-3)(x-4)(x-1)(x+2)(x-2)} = \frac{x^2}{x+2}, \text{ Ans.}$$

8.

$$\frac{x^3-4z}{x^2-5x+6} \cdot \frac{x^2-3x+2}{x^2+2x-3} =$$

$$\frac{x^3-4z}{(x-3)(x-2)} \cdot \frac{(x-2)(x-1)}{(x+3)(x-1)} = \frac{x^3-4z}{x^2-9}, \text{ Ans.}$$

9.

$$\frac{x-7+\frac{12}{x}}{x+3-\frac{18}{x}}=\frac{x^2-7x+12}{x^2+3x-18}=\frac{(x-3)(x-4)}{(x+6)(x-3)}=\frac{x-4}{x+6}.$$

10.

$$\frac{\frac{1}{1-x}-\frac{1}{1+x}}{\frac{1}{1-x}+\frac{1}{1+x}}=\frac{\frac{1+x-1+x}{1-x^2}}{\frac{1+x+1-x}{1-x^2}}=\frac{2x}{2}=x, \text{ Ans.}$$

(393.)

1.

$$13x-20=5x+44$$
$$8x=64; \quad x=8, \text{ Ans.}$$

2.

$$18x+15=24x-15$$
$$30=6x; \quad x=5, \text{ Ans.}$$

3.

$$\frac{3x}{4}+\frac{x-2}{2}-\frac{4x-40}{8}=x-6$$
$$6x+4x-8-4x+40=8x-48$$
$$-2x=-80$$
$$x=40, \text{ Ans.}$$

4.

$$\frac{3x}{b}-\frac{x}{c}=m-c$$
$$3xc-bx=mbc-bc^2$$
$$(3c-b)x=bc(m-c)$$
$$x=\frac{bc(m-c)}{3c-b}, \text{ Ans.}$$

5.

$$a x^2 + b x = m x^2 + n x$$
$$a x - m x = n - b$$
$$x = \frac{n-b}{a-m}, \text{ Ans.}$$

6.

$$.3 x - .02 - .003 x = .7 - .06 x - .006$$
$$300 x - 20 - 3 x = 700 - 60 x - 6$$
$$357 x = 714$$
$$x = 2, \text{ Ans.}$$

7.

$$\frac{3a+x}{x} - 5 - \frac{6}{x} = 0$$
$$3a + x - 5x - 6 = 0$$
$$-4x = 6 - 3a$$
$$x = \frac{3a-6}{4}, \text{ Ans.}$$

8.

$$\frac{1}{a(b-x)} + \frac{1}{b(c-x)} = \frac{1}{a(c-x)}$$
$$bc - bx + ab - ax = b^2 - bx$$
$$-ax = b^2 - ab - bc$$
$$x = \frac{ab + bc - b^2}{a}, \text{ Ans.}$$

9.

$$\frac{a^2 x}{b-c} + de = 3x - \frac{d}{e}$$
$$x \left\{ \frac{a^2}{b-c} - 3 \right\} = -de - \frac{d}{e}$$
$$x \left(\frac{a^2 - 3b + 3c}{b-c} \right) = -\frac{d}{e}(e^2 + 1)$$
$$\therefore x = \frac{d(1+e^2)(c-b)}{e(a^2 - 3b + 3c)}, \text{ Ans.}$$

10.

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$$

$$\frac{x^2-4x+3-x^2+4x-4}{(x-2)(x-3)}=\frac{x^2-12x+35-x^2+12x-36}{(x-6)(x-7)}$$

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}$$

$$x^2-13x+42=x^2-5x+6$$

$$36=8x$$

$$\frac{9}{2}=x=4\tfrac{1}{2},\ \text{Ans.}$$

Verification.

$$\frac{3\frac{1}{2}}{2\frac{1}{2}}-\frac{2\frac{1}{2}}{1\frac{1}{2}}=\frac{-\frac{1}{2}}{-1\frac{1}{2}}-\frac{-1\frac{1}{2}}{-2\frac{1}{2}}$$

$$\frac{7}{5}-\frac{5}{3}=\frac{1}{3}-\frac{3}{5}$$

$$\frac{21-25}{15}=\frac{5-9}{15}$$

$$-\frac{4}{15}=-\frac{4}{15},\ \text{Ans.}$$

(394.)

1.

(a) $\frac{x}{2}-y=1$ (b) $x-\frac{y}{2}=8$

$x-2y=2$

$x=2+2y$

Substitute in (b),

$2x-y=16$

$4+4y-y=16$

$3y=12$

$y=4$

$x=2+8$

$x=10$

2.

$$\frac{x}{3}+\frac{y}{5}=5 \qquad\qquad 2x+\frac{y}{3}=17$$

$$5x+3y=75 \qquad\qquad 6x+y=51$$

$$x=\frac{75-3y}{5}$$

Substituting in 2d equation, we have

$$\begin{aligned} 450-18y+5y&=255\\ 13y&=195\\ \left.\begin{aligned} y&=15\\ x&=6\end{aligned}\right\}&\text{ Ans.}\end{aligned}$$

3.

(1) $x+y+z=12$
(2) $x-y+z=6$
(3) $x-y-z=-2$

From (1) take (2), $2y=6$

$$y=3$$

Sub. in (2) and (3)

$$\begin{aligned} x+z&=9\\ x-z&=1\\ \hline \text{Adding,}\quad 2x&=10;\quad x=5\\ \text{Subtracting,}\quad 2z&=8\\ z&=4\end{aligned}$$

$x=5,\quad y=3,\quad z=4,$ Ans.

4.

(1) $\dfrac{7x-9y}{3}=7$

(2) $\dfrac{5x+5y}{19}=5$

Clearing of fractions,

$$\begin{aligned} (1)\quad 7x-\ \ 9y&=21\\ (2)\quad x+\ \ \ y&=19\\ \hline 7x-\ \ 9y&=21\\ 7x+\ \ 7y&=133\\ \hline -16y&=-112\\ y&=7\\ x&=12, \text{ from (2)}\end{aligned}$$

5.

Clear of fractions,

$$
\begin{array}{lrl}
(1) & 3x + y - z &= 76 \\
(2) & x + y + 6z &= 63 \\
(3) & -6x + y &= -133 \\
\hline
(1)-(3) \text{ gives } & 9x - z &= 209 \\
(2)-(3) \text{ gives } & 7x + 6z &= 196 \\
\hline
& 54x - 6z &= 1254 \\
& 7x + 6z &= 196 \\
\hline
& 61x &= 1450 \\
& x &= \frac{1450}{61}
\end{array}
$$

From (3), $-\frac{8700}{61} + y = -133$

$$-8700 + 61y = -8113$$

$$61y = 587$$

$$y = \frac{587}{61}$$

From (1), $\frac{4350}{61} + \frac{587}{61} - z = 76$

$$z = -\frac{301}{61}$$

$x = 23\frac{47}{61}$, $y = 9\frac{38}{61}$, $z = -4\frac{57}{61}$, Ans.

6.

$$\frac{a}{x} + \frac{b}{y} = m \quad \therefore \frac{ca}{x} + \frac{cb}{y} = cm$$

$$\frac{c}{x} + \frac{d}{y} = n \quad \therefore \frac{ca}{x} + \frac{da}{y} = na$$

$$\frac{cb - da}{y} = cm - na \qquad y = \frac{cb - da}{cm - na}$$

Again,

$$\left.\begin{array}{l} \frac{da}{x} + \frac{bd}{y} = dm \\ \frac{bc}{x} + \frac{bd}{y} = bn \end{array}\right\} \frac{da}{x} - \frac{bc}{x} = dm - bn$$

$$da - bc = x(dm - bn)$$

$$x = \frac{da - bc}{dm - bn}$$

Proof.

$$\frac{\frac{a}{da-bc}}{dm-bn}+\frac{\frac{b}{cb-da}}{cm-na} \text{ should} = m$$

$$\frac{adm-abn}{da-bc}+\frac{bcm-bna}{cb-da}$$

$$=\frac{adm-abn-bcm+bna}{da-bc}=m.$$

7.

$$\frac{x}{7}-\frac{y}{10}=-20$$

$$\frac{x}{4}+3y=134$$

(1) $10x - 7y = -1400$

(2) $x + 12y = 536$

(1) $x = -140 + .7y$

(2) $x = 536 - 12y$

(2) — (1) gives $0 = 676 - 12.7y$

$$y=\frac{6760}{127}$$

(2) $x = 536 - 12\left(\frac{6760}{127}\right)$

$$x=\frac{68072}{127}-\frac{81120}{127}=-\frac{13048}{127}$$

$$y=\frac{6760}{127}=53\tfrac{29}{127}$$

$$x=-\frac{13048}{127}=-102\tfrac{94}{127}$$

8.

$$x - ay + a^2z = a^3 \qquad (1)$$
$$x - by + b^2z = b^3 \qquad (2)$$
$$x - cy + c^2z = c^3 \qquad (3)$$

(1) — (2) gives $(b - a)y + (a^2 - b^2)z = a^3 - b^3$ (A)

(2) — (3) gives $(c - b)y + (b^2 - c^2)z = b^3 - c^3$ (B)

(A) gives $y - (a + b)z = -a^2 - ab - b^2$

(B) gives $y - (c + b)z = -b^2 - bc - c^2$

(A) — (B) gives $(c - a)z = c^2 + bc - a^2 - ab$

$$= (c^2 - a^2) + b(c - a)$$
$$z = c + a + b$$

(A) $y = (a + b)z - a^2 - ab - b^2$

$$y = c(a + b) + (a + b)^2 - a^2 - ab - b^2$$
$$= ca + bc + a^2 + 2ab + b^2 - a^2 - ab - b^2$$
$$= ca + bc + ab$$

(1) gives $x = ay - a^2z + a^3$

$$x = a^2c + abc + a^2b - a^2c - a^3 - a^2b + a^3$$

$$\left.\begin{aligned} x &= abc \\ y &= ca + bc + ab \\ z &= c + a + b \end{aligned}\right\} \text{Ans.}$$

Substitute in (3) and prove.

9.

$$\begin{array}{lrcl} (1) & 8x - 4y + z &=& 24 \\ (2) & 6x + y - z &=& 84 \\ (3) & x - 3y - 4z &=& -80 \end{array}$$

(1) + (2) gives $14x - 3y = 108$ (A)

(2) $\times$ 4 gives $24x + 4y - 4z = 336$

(3) gives $x - 3y - 4z = -80$

Subtracting, $23x + 7y = 416$ (B)

Combine (A) and (B)

$$\begin{aligned} 98x - 21y &= 756 \\ 69x + 21y &= 1248 \\ \hline 167x \quad &= 2004 \\ x &= 12 \end{aligned}$$

$$\text{(A) gives } 168 - 3y = 108$$
$$y = 20$$
$$\text{(1) gives } 96 - 80 + z = 24$$
$$z = 8$$
$$x = 12, \quad y = 20, \quad z = 8, \text{ Ans.}$$

10.

$$\begin{aligned} a &= y + z \quad (1) \\ b &= x + z \quad (2) \\ c &= x + y \quad (3) \\ \hline a - b &= y - x \quad (A) \\ c &= y + x \quad (3) \\ \hline \frac{a-b+c}{2} &= y \end{aligned}$$

Add (A) and (3) for y.
Subtract (3) from (A) gives

$$x = \frac{c - a + b}{2}$$

Add (1) and (2) and we have

$$a + b = x + y + 2z; \quad \text{but } x + y = c$$

therefore, by substitution, $z = \dfrac{a + b - c}{2}$, Ans.

(395.)

1.

$x =$ her age.

Statement.

$$x - \frac{x}{2} - \frac{1}{2}\left(x - \frac{x}{2}\right) = 19$$
$$\frac{x}{4} = 19$$
$$x = 76, \text{ Ans.}$$

2.

$$x = \text{number of apple-trees.}$$
$$3x = \text{" " pear-trees.}$$
$$4(x-4) = 3x - 4$$
$$4x - 16 = 3x - 4$$
$$\left.\begin{array}{r} x = 12 \text{ apple-trees,} \\ 3x = 36 \text{ pear-trees,} \end{array}\right\} \text{Ans.}$$

3.

$$\left.\begin{array}{ll} (1) & \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 \\ (2) & \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 \\ (3) & \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38 \end{array}\right\}$$

Multiply each equation by the denominator of x.

$$(1) \quad x+\frac{2y}{3}+\frac{z}{2} = 124$$
$$(2) \quad x+\frac{3}{4}y+\frac{3}{5}z = 141$$
$$(3) \quad x+\frac{4}{5}y+\frac{2}{3}z = 152$$

$(2)-(1)$ gives $\frac{1}{12}y+\frac{1}{10}z=17 \quad (A)$

$(3)-(2)$ gives $\frac{1}{20}y+\frac{1}{15}z=11 \quad (B)$

$12\times(A)$ gives $y+\frac{6}{5}z=204$

$20\times(B)$ gives $y+\frac{20}{15}z=220$

Subtracting,

$$\frac{2}{15}z=16$$

By substitution, $\left\{\begin{array}{l} z = 120 \\ y = 60 \\ x = 24 \end{array}\right\}$ Ans.

4.

$x =$ sum left to each.

By conditions,

$4x =$ what eldest had after seven years.

$x - 1000 =$ " youngest " "

$$4x = 5x - 5000$$

$$x = \$5000, \text{ Ans.}$$

5.

$x =$ number of men, $y =$ number of widows.

$$(1) \quad 3x + 3y = 111$$

$$(2) \quad 2\tfrac{1}{6}x + 3\tfrac{1}{2}y = \$110 - .50 = 109.5$$

$$\begin{aligned} (2) \text{ gives } 13x + 21y &= 657 \\ 7 \,.\, (1) \text{ gives } 21x + 21y &= 777 \\ \hline 8x \qquad\quad &= 120 \end{aligned}$$

$$\left.\begin{aligned} x &= 15 \text{ men,} \\ y &= 22 \text{ women,} \end{aligned}\right\} \text{Ans.}$$

6.

$x =$ number of cubic inches of copper.

$y =$ " " " " tin.

By conditions we have,

$$(1) \quad x + y = 100$$

$$(2) \quad 5\tfrac{1}{4}x + 4\tfrac{1}{2}y = 505$$

$$\begin{aligned} 21x + 18y &= 2020 \\ 18x + 18y &= 1800 \\ \hline 3x \qquad\quad &= 220 \end{aligned}$$

$$x = 73\tfrac{1}{3}$$

$$y = 26\tfrac{2}{3}$$

$$\begin{aligned} 5\tfrac{1}{4} \times 73\tfrac{1}{3} &= 385, \text{ copper.} \\ 4\tfrac{1}{2} \times 26\tfrac{2}{3} &= 120, \text{ tin.} \\ \hline &505, \text{ proof.} \end{aligned}$$

7.

$$\begin{array}{ll} (1) & x + 220 = y + z \\ (2) & y + 220 = 2x + 2z \\ (3) & z + 220 = 3x + 3y \end{array}$$

$$\begin{array}{lrr} & x - y - z = -220 & (1) \\ & -2x + y - 2z = -220 & (2) \\ & -3x - 3y + z = -220 & (3) \\ \text{Multiply (1) by 2,} & 2x - 2y - 2z = -440 & (4) \\ \text{Add (1) and (3),} & -2x - 4y = -440 & (5) \\ \text{Subtract (2) from (4),} & 4x - 3y = -220 & (6) \\ \text{Multiply (5) by 2,} & -4x - 8y = -880 & (7) \\ \text{Add (6) and (7),} & -11y = -1100 & \\ \text{Whence,} & y = 100 & \\ \text{Substitute in (6),} & x = 20 & \text{Ans.} \\ \text{Substitute in (1),} & z = 140 & \end{array}$$

8.

For every day idle he loses not only the wage of that day but also the forfeit.

Let $x =$ number of days idle.

$1.75x =$ amount lost.

$36 - x =$ number of days he worked.

$36 \times 1.25 =$ sum he should receive.

Or, $\$45 - 1.75x = 17$

$$-\frac{7}{4}x = -28$$

$$\left.\begin{array}{r} x = 16 \text{ days idle,} \\ 36 - x = 20 \text{ days of work,} \end{array}\right\} \text{Ans.}$$

9.

$x =$ number of days in which A can do the work.

$y =$ " " " " B " " "

$\frac{1}{x} =$ part A does in one day.

$\frac{1}{y} =$ " B " " "

In 4 days they do $\frac{4}{10}$ of it $= \frac{2}{5}$.

$$\frac{4}{x} + \frac{4}{y} = \frac{2}{5}$$

B does the rest, $\frac{16}{y} = \frac{3}{5}$

$$\frac{4}{y} = \frac{3}{20}$$

$$\frac{4}{x} = \frac{2}{5} - \frac{3}{20} = \frac{1}{4}$$

$$\left.\begin{array}{l} x = 16, \text{ A's time,} \\ y = 26\frac{2}{3}, \text{ B's time,} \end{array}\right\} \text{Ans.}$$

10.

$$\begin{aligned}
d &= \text{common difference.} \\
y &= \text{units figure.} \\
d + y &= \text{tens figure.} \\
2d + y &= \text{hundreds figure.} \\
3d + 3y \text{ or } 3(d + y) &= \text{sum of digits.} \\
100(2d + y) + 10(d + y) + y &= \text{number.} \\
200d + 100y + 10d + 10y + y &= \quad \text{"} \\
210d + 111y &= \quad \text{"} \\
100y + 10d + 10y + 2d + y &= \quad \text{"} \quad \text{reversed.} \\
111y + 12d &= \quad \text{"} \quad\quad \text{"}
\end{aligned}$$

Statement.

(1) $$\frac{210d + 111y}{\frac{3d + 3y}{2}} = 41$$

(2) $$396 + 210d + 111y = 111y + 12d$$

(1) simplified gives $140d + 74y = 41(d + y)$

(2) " gives $-198d = 396$

$$\begin{aligned}
\text{(2) gives } d &= -2 \\
\text{(1) gives } -280 + 74y &= -82 + 41y \\
33y &= 198 \\
y &= 6 \\
d + y &= 4 \\
2d + y &= 2 \\
\text{Number} &= 246, \text{ Ans.}
\end{aligned}$$

11.

$na =$ distance travelled by 1st before 2d starts.

$x =$ number of days 2d goes before overtaking 1st.

$ax =$ distance 1st travels in x days.

$na + ax =$ distance to be travelled by 2d.

$$\frac{na + ax}{b} = x$$
$$na + ax = bx$$
$$na = (b - a)\,x$$
$$x = \frac{na}{b - a},\ \text{Ans.}$$

12.

$x =$ length. $y =$ width. $xy =$ area.

$$(1) \qquad (x + 5)(y + 4) = xy + 240$$
$$xy + 5y + 4x + 20 = xy + 240$$
$$5y + 4x = 220$$
$$(2) \qquad (x - 4)(y - 5) = xy - 210$$
$$xy - 4y - 5x + 20 = xy - 210$$
$$4y + 5x = 230$$

$$\begin{aligned} 20y + 16x &= 880 \\ 20y + 25x &= 1150 \\ \hline 9x &= 270 \\ x &= 30 \\ y &= 20 \\ xy &= 600 \end{aligned}$$

$$\left.\begin{aligned} \text{Length} &= 30 \text{ rods,} \\ \text{Width} &= 20 \text{ "} \\ \text{Area} &= 600 \text{ sq. rods,} \end{aligned}\right\} \text{Ans.}$$

(396.)

1.

$$\left(\frac{4m}{3c}\right)^3 = \frac{64m^3}{27c^3},\ \text{Ans.} \qquad \left(\frac{3ax}{4c^2y}\right)^3 = \frac{27a^3x^3}{64c^6y^3},\ \text{Ans.}$$

2.

$$\left.\begin{aligned}(a+2x)^2 &= a^2+4ax+4x^2\\ (ax+b)^2 &= a^2x^2+2abx+b^2\end{aligned}\right\}\text{ Ans.}$$

3.

$$\begin{array}{r|l} & a^2+4ax+4x^2 \;\underline{|\, a+2x}, \text{ Ans.}\\ & a^2 \\ \hline \underline{2a+2x} & 4ax+4x^2 \\ & \underline{4ax+4x^2} \end{array}$$

$$\begin{array}{r|l} & a^2x^2+2abx+b^2 \;\underline{|\, ax+b}, \text{ Ans.}\\ & a^2x^2 \\ \hline \underline{2ax+b} & 2abx+b^2 \\ & \underline{2abx+b^2} \end{array}$$

4.

$$\sqrt[3]{-64a^6y^3} = -4a^2y, \text{ Ans.} \qquad \sqrt[3]{125a^3b^6y^9} = 5ab^2y^3, \text{ Ans.}$$

5.

$$(a+x)^4 = a^4+4a^3x+6a^2x^2+4ax^3+x^4$$

$$(2x+3y)^3 = 8x^3+36x^2y+54xy^2+27y^3$$

$$(x^2-z^3)^5 = x^{10}-5x^8z^3+10x^6z^6-10x^4z^9+5x^2z^{12}-z^{15}$$

$$(x+y)^{\frac{1}{2}} = x^{\frac{1}{2}}\left(1+\frac{y}{2x}-\frac{y^2}{2.4x^2}+\frac{3y^3}{2.4.6x^3}-\frac{3.5y^4}{2.4.6.8x^4}+\dots\right)$$

$$(a+b)^n = a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2$$

$$+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\dots$$

$$(a-c)^{-2} = \frac{1}{a^2}\left(1-\frac{c}{a}\right)^{-2}$$

$$= \frac{1}{a^2}\left(1-\frac{2c}{a}+\frac{3c^2}{a^2}-\frac{4c^3}{a^3}+\frac{5c^4}{a^4}+\dots\right)$$

10.

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$$

$$\frac{x^2-4x+3-x^2+4x-4}{(x-2)(x-3)}=\frac{x^2-12x+35-x^2+12x-36}{(x-6)(x-7)}$$

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}$$

$$x^2-13x+42=x^2-5x+6$$

$$36=8x$$

$$\frac{9}{2}=x=4\tfrac{1}{2},\text{ Ans.}$$

Verification.

$$\frac{3\frac{1}{2}}{2\frac{1}{2}}-\frac{2\frac{1}{2}}{1\frac{1}{2}}=\frac{-\frac{1}{2}}{-1\frac{1}{2}}-\frac{-1\frac{1}{2}}{-2\frac{1}{2}}$$

$$\frac{7}{5}-\frac{5}{3}=\frac{1}{3}-\frac{3}{5}$$

$$\frac{21-25}{15}=\frac{5-9}{15}$$

$$-\frac{4}{15}=-\frac{4}{15},\text{ Ans.}$$

(394.)

1.

(a) $\frac{x}{2}-y=1$

$x-2y=2$

$x=2+2y$

Substitute in (b),

$x=2+8$

$x=10$

(b) $x-\frac{y}{2}=8$

$2x-y=16$

$4+4y-y=16$

$3y=12$

$y=4$

2.

$$\frac{x}{3}+\frac{y}{5}=5 \qquad 2x+\frac{y}{3}=17$$

$$5x+3y=75 \qquad 6x+y=51$$

$$x=\frac{75-3y}{5}$$

Substituting in 2d equation, we have

$$\begin{aligned} 450-18y+5y&=255\\ 13y&=195\\ y&=15\\ x&=6 \end{aligned}\Bigg\}\text{ Ans.}$$

3.

$$\begin{aligned} &(1)\ x+y+z=12\\ &(2)\ x-y+z=6\\ &(3)\ x-y-z=-2 \end{aligned}$$

From (1) take (2), $2y=6$

$$y=3$$

Sub. in (2) and (3)

$$\begin{aligned} x+z&=9\\ x-z&=1 \end{aligned}$$

Adding, $2x=10;\quad x=5$

Subtracting, $2z=8$

$$z=4$$

$$x=5,\quad y=3,\quad z=4,\text{ Ans.}$$

4.

$$(1)\ \frac{7x-9y}{3}=7$$

$$(2)\ \frac{5x+5y}{19}=5$$

Clearing of fractions,

$$\begin{aligned} &(1)\ 7x-\ \ 9y=21\\ &(2)\ \ \ x+\ \ \ y=19\\ &\quad 7x-\ \ 9y=21\\ &\quad 7x+\ \ 7y=133\\ &\qquad\ -16y=-112\\ &\qquad\qquad y=7\\ &\qquad\qquad x=12,\text{ from (2)} \end{aligned}$$

5.

Clear of fractions,

$$
\begin{array}{ll}
(1) & 3x + y - z = 76 \\
(2) & x + y + 6z = 63 \\
(3) & -6x + y = -133 \\
\hline
\end{array}
$$

$$
\begin{aligned}
(1)-(3) \text{ gives } 9x - z &= 209 \\
(2)-(3) \text{ gives } 7x + 6z &= 196 \\
54x - 6z &= 1254 \\
7x + 6z &= 196 \\
61x &= 1450 \\
x &= \frac{1450}{61}
\end{aligned}
$$

From (3),
$$
\begin{aligned}
-\frac{8700}{61} + y &= -133 \\
-8700 + 61y &= -8113 \\
61y &= 587 \\
y &= \frac{587}{61}
\end{aligned}
$$

From (1),
$$
\begin{aligned}
\frac{4350}{61} + \frac{587}{61} - z &= 76 \\
z &= -\frac{301}{61}
\end{aligned}
$$

$x = 23\frac{47}{61}$, $y = 9\frac{38}{61}$, $z = -4\frac{57}{61}$, Ans.

6.

$$
\frac{a}{x} + \frac{b}{y} = m \quad \therefore \frac{ca}{x} + \frac{cb}{y} = cm
$$

$$
\frac{c}{x} + \frac{d}{y} = n \quad \therefore \frac{ca}{x} + \frac{da}{y} = na
$$

$$
\frac{cb - da}{y} = cm - na \qquad y = \frac{cb - da}{cm - na}
$$

Again,

$$
\left.\begin{aligned}
\frac{da}{x} + \frac{bd}{y} &= dm \\
\frac{bc}{x} + \frac{bd}{y} &= bn
\end{aligned}\right\} \frac{da}{x} - \frac{bc}{x} = dm - bn
$$

$$
\begin{aligned}
da - bc &= x(dm - bn) \\
x &= \frac{da - bc}{dm - bn}
\end{aligned}
$$

Proof.

$$\frac{a}{\dfrac{da-bc}{dm-bn}}+\frac{b}{\dfrac{cb-da}{cm-na}} \text{ should} = m$$

$$\frac{adm-abn}{da-bc}+\frac{bcm-bna}{cb-da}$$

$$=\frac{adm-abn-bcm+bna}{da-bc}=m.$$

7.

$$\frac{x}{7}-\frac{y}{10}=-20$$

$$\frac{x}{4}+3y=134$$

$$\begin{array}{ll}
(1) & 10x-\ \ 7y=-1400 \\
(2) & \quad x+12y=536 \\
\hline
(1) & \quad x=-140+.7y \\
(2) & \quad x=536-12y
\end{array}$$

$$(2)-(1) \text{ gives } 0=676-12.7y$$

$$y=\frac{6760}{127}$$

$$(2) \qquad x=536-12\left(\frac{6760}{127}\right)$$

$$x=\frac{68072}{127}-\frac{81120}{127}=-\frac{13048}{127}$$

$$y=\frac{6760}{127}=53\tfrac{29}{127}$$

$$x=-\frac{13048}{127}=-102\tfrac{94}{127}$$

12.

$$\frac{2a}{a^2+a-6}, \frac{4c}{a^2-4} = \frac{2a}{(a+3)(a-2)}, \frac{4c}{(a-2)(a+2)} =$$

$$\frac{2a(a+2)}{(a+3)(a-2)(a+2)}, \frac{4c(a+3)}{(a+3)(a-2)(a+2)}, \text{ Ans.}$$

(391.)

1.

$$\frac{2a}{a+x} + \frac{3x}{a-x} + \frac{3x^2+a^2}{a^2-x^2}$$

$$= \frac{2a(a-x)+3x(a+x)+3x^2+a^2}{a^2-x^2}$$

$$= \frac{2a^2-2ax+3ax+3x^2+3x^2+a^2}{a^2-x^2}$$

$$= \frac{3a^2+ax+6x^2}{a^2-x^2}, \text{ Ans.}$$

2.

$$\frac{7}{5a+4b} - \frac{7}{5a-4b} = \frac{7(5a-4b)-7(5a+4b)}{25a^2-16b^2}$$

$$= \frac{-56b}{25a^2-16b^2}, \text{ Ans.}$$

3.

$$\frac{a^2+2ax+x^2}{a-x} - \frac{a^2-2ax+x^2}{a+x}$$

$$= \frac{(a+x)^2(a+x)-(a-x)^2(a-x)}{a^2-x^2}$$

$$= \frac{(a^3+3a^2x+3ax^2+x^3)-(a^3-3a^2x+3ax^2-x^3)}{a^2-x^2}$$

$$= \frac{6a^2x+2x^3}{a^2-x^2}, \text{ Ans.}$$

4.

$$\left(5a - \frac{x-7}{4}\right) - \left(2a + \frac{x-3}{12}\right) = 3a - \frac{x-7}{4} - \frac{x-3}{12}$$

$$= \frac{36a - 3x + 21 - x + 3}{12} = \frac{36a - 4x + 24}{12}, \text{ Ans.}$$

5.

$$\frac{3+2y}{2-y} + \frac{16y - y^2}{y^2-4} - \frac{2-3y}{2+y}$$

$$= \frac{(3+2y)(-y-2) + (16y - y^2) - (2-3y)(y-2)}{y^2-4}$$

$$= \frac{-3y - 6 - 2y^2 - 4y + 16y - y^2 - 2y + 4 + 3y^2 - 6y}{y^2-4}$$

$$= \frac{y-2}{y^2-4} = \frac{1}{y+2}, \text{ Ans.}$$

6.

$$\frac{x}{1+x} - \frac{x}{1-x} + \frac{x^2}{x^2-1} = \frac{x(1-x) - x(1+x) - x^2}{1-x^2}$$

$$= \frac{x - x^2 - x - x^2 - x^2}{1-x^2} = \frac{-3x^2}{1-x^2}, \text{ Ans.}$$

7.

$$\frac{2x-6}{x^2+3x+2} - \frac{x+2}{x^2-2x-3} - \frac{x+1}{x^2-x-6}$$

$$= \frac{2x-6}{(x+2)(x+1)} - \frac{x+2}{(x-3)(x+1)} - \frac{x+1}{(x-3)(x+2)}$$

$$= \frac{2(x-3)^2 - (x+2)(x+2) - (x+1)^2}{(x+1)(x+2)(x-3)}$$

$$= \frac{2x^2 - 12x + 18 - x^2 - 4x - 4 - x^2 - 2x - 1}{(x+1)(x+2)(x-3)}$$

$$= \frac{-18x + 13}{(x+1)(x+2)(x-3)}, \text{ Ans.}$$

8.

$$\frac{5a+1}{3a+3}+\frac{3a-1}{2-2a}=\frac{5a+1}{3(a+1)}-\frac{3a-1}{2(a-1)}$$

$$=\frac{2(5a+1)(a-1)-3(a+1)(3a-1)}{6(a^2-1)}$$

$$=\frac{10a^2-8a-2-9a^2-6a+3}{6(a^2-1)}=\frac{a^2-14a+1}{6(a^2-1)},\ \text{Ans.}$$

9.

$$\frac{3a+1}{12a}-\frac{2b-1}{8b}+\frac{4c-1}{16c}-\frac{6d+1}{24d}$$

$$=\frac{4bcd(3a+1)-6acd(2b-1)+3abd(4c-1)-2abc(6d+1)}{48abcd}$$

$$=\frac{12abcd+4bcd-12abcd+6acd+12abcd-8abd-12abcd-2abc}{48abcd}$$

$$=\frac{4bcd+6acd-3abd-2abc}{48abcd},\ \text{Ans.}$$

10.

$$\frac{1}{(a-b)(b-c)}+\frac{1}{(b-a)(a-c)}-\frac{1}{(c-a)(c-b)}$$

$$=\frac{a-c+c-b-a+b}{(a-b)(a-c)(b-c)}=0,\ \text{Ans.}$$

(392.)

1.

$$\left(\frac{a^2-x^2}{a}\right)\frac{(a^2+x^2)}{ax}\,\frac{a^2x^2}{y^2}=\frac{x(a^4-x^4)}{y^2},\ \text{Ans.}$$

2.

$$\frac{4x}{9}\div\frac{4x^2}{45}=\frac{5}{x};\quad \frac{2x}{21a^2}\div\frac{2}{ax^2}=\frac{ax^3}{21a^2}=\frac{x^3}{21a},\ \text{Ans.}$$

3.

$$\frac{3x^2}{a^3+x^3} \times \frac{a+x}{x} = \frac{3x}{a^2-ax+x^2}; \quad \frac{a+x}{c+d} \times \frac{c^2-d^2}{(a+x)^2} = \frac{c-d}{a+x}.$$

4.

$$\left(3a + \frac{4x^2}{5c}\right)\left(3a - \frac{4x^2}{5c}\right) = 9a^2 - \frac{16x^4}{25c^2}, \text{ Ans.}$$

5.

$$\frac{n+\frac{1}{n}}{\frac{a}{n}+\frac{1}{n}} = \frac{\frac{n^2+1}{n}}{\frac{a+1}{n}} = \frac{n^2+1}{a+1}, \text{ Ans.}$$

6.

$$\frac{a^{-2}b^{-3}}{a^{-2}e^{-1}} = \frac{a^2 e}{a^2 b^3} = \frac{e}{b^3}, \text{ Ans.}$$

$$\frac{x^2+y^{-3}}{mn^{-4}e^{-2}} = \frac{\left(x^2+\frac{1}{y^3}\right)n^4e^2}{m} = \frac{n^4e^2(x^2y^3+1)}{my^3}, \text{ Ans.}$$

7.

$$\left(\frac{x^2-3x+2}{x^2-8x+15}\right)\frac{(x^2-7x+12)}{(x^2-5x+4)}\left(\frac{x^3-5x^2}{x^2-4}\right) =$$

$$\frac{(x-2)(x-1)(x-4)(x-3)x^2(x-5)}{(x-5)(x-3)(x-4)(x-1)(x+2)(x-2)} = \frac{x^2}{x+2}, \text{ Ans.}$$

8.

$$\frac{x^3-4z}{x^2-5x+6} \cdot \frac{x^2-3x+2}{x^2+2x-3} =$$

$$\frac{x^3-4z}{(x-3)(x-2)} \cdot \frac{(x-2)(x-1)}{(x+3)(x-1)} = \frac{x^3-4z}{x^2-9}, \text{ Ans.}$$

10.

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$$

$$\frac{x^2-4x+3-x^2+4x-4}{(x-2)(x-3)}=\frac{x^2-12x+35-x^2+12x-36}{(x-6)(x-7)}$$

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}$$

$$x^2-13x+42=x^2-5x+6$$

$$36=8x$$

$$\frac{9}{2}=x=4\tfrac{1}{2},\text{ Ans.}$$

Verification.

$$\frac{3\frac{1}{2}}{2\frac{1}{2}}-\frac{2\frac{1}{2}}{1\frac{1}{2}}=\frac{-\frac{1}{2}}{-1\frac{1}{2}}-\frac{-1\frac{1}{2}}{-2\frac{1}{2}}$$

$$\frac{7}{5}-\frac{5}{3}=\frac{1}{3}-\frac{3}{5}$$

$$\frac{21-25}{15}=\frac{5-9}{15}$$

$$-\frac{4}{15}=-\frac{4}{15},\text{ Ans.}$$

(394.)

1.

(a) $\frac{x}{2}-y=1$

$x-2y=2$

$x=2+2y$

Substitute in (b),

$x=2+8$

$x=10$

(b) $x-\frac{y}{2}=8$

$2x-y=16$

$4+4y-y=16$

$3y=12$

$y=4$

2.

$$\frac{x}{3}+\frac{y}{5}=5 \qquad\qquad 2x+\frac{y}{3}=17$$

$$5x+3y=75 \qquad\qquad 6x+y=51$$

$$x=\frac{75-3y}{5}$$

Substituting in 2d equation, we have

$$\begin{aligned} 450-18y+5y&=255\\ 13y&=195\\ y&=15\\ x&=6 \end{aligned}\Bigg\}\text{ Ans.}$$

3.

(1) $x+y+z=12$

(2) $x-y+z=6$

(3) $x-y-z=-2$

From (1) take (2), $2y=6$

$y=3$

Sub. in (2) and (3)

$$\begin{aligned} x+z&=9\\ x-z&=1 \end{aligned}$$

Adding, $2x=10;\quad x=5$

Subtracting, $2z=8$

$z=4$

$x=5,\quad y=3,\quad z=4,$ Ans.

4.

(1) $\dfrac{7x-9y}{3}=7$

(2) $\dfrac{5x+5y}{19}=5$

Clearing of fractions,

$$\begin{aligned} (1)\quad 7x-9y&=21\\ (2)\quad x+y&=19\\ \hline 7x-9y&=21\\ 7x+7y&=133\\ \hline -16y&=-112\\ y&=7\\ x&=12,\text{ from (2)} \end{aligned}$$

5.

Clear of fractions,

$$\begin{aligned}
&(1) & 3x + y - z &= 76\\
&(2) & x + y + 6z &= 63\\
&(3) & -6x + y \quad &= -133\\
&(1)-(3) \text{ gives} & 9x - z &= 209\\
&(2)-(3) \text{ gives} & 7x + 6z &= 196\\
&& 54x - 6z &= 1254\\
&& 7x + 6z &= 196\\
&& 61x &= 1450\\
&& x &= \frac{1450}{61}
\end{aligned}$$

From (3),

$$\begin{aligned}
-\frac{8700}{61} + y &= -133\\
-8700 + 61y &= -8113\\
61y &= 587\\
y &= \frac{587}{61}
\end{aligned}$$

From (1),

$$\begin{aligned}
\frac{4350}{61} + \frac{587}{61} - z &= 76\\
z &= -\frac{301}{61}
\end{aligned}$$

$$x = 23\tfrac{47}{61}, \quad y = 9\tfrac{38}{61}, \quad z = -4\tfrac{57}{61}, \text{ Ans.}$$

6.

$$\frac{a}{x} + \frac{b}{y} = m \qquad \therefore \frac{ca}{x} + \frac{cb}{y} = cm$$

$$\frac{c}{x} + \frac{d}{y} = n \qquad \therefore \frac{ca}{x} + \frac{da}{y} = na$$

$$\frac{cb - da}{y} = cm - na \qquad y = \frac{cb - da}{cm - na}$$

Again,

$$\left.\begin{aligned}
\frac{da}{x} + \frac{bd}{y} &= dm\\
\frac{bc}{x} + \frac{bd}{y} &= bn
\end{aligned}\right\} \frac{da}{x} - \frac{bc}{x} = dm - bn$$

$$da - bc = x(dm - bn)$$

$$x = \frac{da - bc}{dm - bn}$$

Proof.

$$\frac{\dfrac{a}{da-bc}}{dm-bn}+\frac{\dfrac{b}{cb-da}}{cm-na} \text{ should} = m$$

$$\frac{adm-abn}{da-bc}+\frac{bcm-bna}{cb-da}$$

$$=\frac{adm-abn-bcm+bna}{da-bc}=m.$$

7.

$$\frac{x}{7}-\frac{y}{10}=-20$$

$$\frac{x}{4}+3y=134$$

(1) $10x - 7y = -1400$

(2) $x + 12y = 536$

(1) $x = -140 + .7y$

(2) $x = 536 - 12y$

(2) — (1) gives $0 = 676 - 12.7y$

$$y=\frac{6760}{127}$$

(2) $x = 536 - 12\left(\dfrac{6760}{127}\right)$

$$x=\frac{68072}{127}-\frac{81120}{127}=-\frac{13048}{127}$$

$$y=\frac{6760}{127}=53\tfrac{29}{127}$$

$$x=-\frac{13048}{127}=-102\tfrac{94}{127}$$

8.

$$x - ay + a^2z = a^3 \qquad (1)$$
$$x - by + b^2z = b^3 \qquad (2)$$
$$x - cy + c^2z = c^3 \qquad (3)$$

(1) — (2) gives $(b - a)y + (a^2 - b^2)z = a^3 - b^3$ (A)
(2) — (3) gives $(c - b)y + (b^2 - c^2)z = b^3 - c^3$ (B)

(A) gives $y - (a + b)z = -a^2 - ab - b^2$
(B) gives $y - (c + b)z = -b^2 - bc - c^2$

(A) — (B) gives $(c - a)z = c^2 + bc - a^2 - ab$
$$= (c^2 - a^2) + b(c - a)$$
$$z = c + a + b$$

(A) $y = (a + b)z - a^2 - ab - b^2$
$$y = c(a + b) + (a + b)^2 - a^2 - ab - b^2$$
$$= ca + bc + a^2 + 2ab + b^2 - a^2 - ab - b^2$$
$$= ca + bc + ab$$

(1) gives $x = ay - a^2z + a^3$
$$x = a^2c + abc + a^2b - a^2c - a^3 - a^2b + a^3$$
$$\left.\begin{aligned} x &= abc \\ y &= ca + bc + ab \\ z &= c + a + b \end{aligned}\right\} \text{Ans.}$$

Substitute in (3) and prove.

9.

(1) $8x - 4y + z = 24$
(2) $6x + y - z = 84$
(3) $x - 3y - 4z = -80$

(1) + (2) gives $14x - 3y = 108$ (A)

(2) × 4 gives $24x + 4y - 4z = 336$
(3) gives $x - 3y - 4z = -80$
Subtracting, $23x + 7y = 416$ (B)

Combine (A) and (B)

$$\begin{aligned} 98x - 21y &= 756 \\ 69x + 21y &= 1248 \\ \hline 167x &= 2004 \\ x &= 12 \end{aligned}$$

$$\text{(A) gives } 168 - 3y = 108$$
$$y = 20$$
$$\text{(1) gives } 96 - 80 + z = 24$$
$$z = 8$$
$$x = 12, \quad y = 20, \quad z = 8, \text{ Ans.}$$

10.

$$\begin{aligned} a &= y + z \quad (1) \\ b &= x + z \quad (2) \\ c &= x + y \quad (3) \\ \hline a - b &= y - x \quad (A) \\ c &= y + x \quad (3) \\ \hline \frac{a-b+c}{2} &= y \end{aligned}$$

Add (A) and (3) for y.
Subtract (3) from (A) gives

$$x = \frac{c - a + b}{2}$$

Add (1) and (2) and we have

$$a + b = x + y + 2z; \text{ but } x + y = c$$

therefore, by substitution, $z = \dfrac{a + b - c}{2}$, Ans.

(395.)

1.

$x =$ her age.

Statement.

$$x - \frac{x}{2} - \frac{1}{2}\left(x - \frac{x}{2}\right) = 19$$
$$\frac{x}{4} = 19$$
$$x = 76, \text{ Ans.}$$

6.

$$\begin{array}{ll} (1) & 2x^2 + xy - 5y^2 = 20 \\ (2) & 2x - 3y = 1 \end{array}$$

From (2), $$x = \frac{1+3y}{2}$$

$$2\left(\frac{1+3y}{2}\right)^2 + \left(\frac{1+3y}{2}\right)y - 5y^2 = 20$$

$$1 + 6y + 9y^2 + y + 3y^2 - 10y^2 = 40$$

$$2y^2 + 7y = 39$$

$$y^2 + \frac{7}{2}y = \frac{39}{2}$$

Completing the square,

$$y^2 + \frac{7}{2}y + \frac{49}{16} = \frac{39}{2} + \frac{49}{16}$$

$$= \frac{312+49}{16} = \frac{361}{16}$$

Extracting root,

$$y + \frac{7}{4} = \pm\frac{19}{4}$$

$$y = \frac{19}{4} - \frac{7}{4} = 3$$

or $$= -\frac{19}{4} - \frac{7}{4} = -\frac{26}{4} = -\frac{13}{2}$$

$$\left.\begin{array}{l} y = 3, \text{ or } -\frac{13}{2} \\ x = 5, \text{ or } -\frac{37}{4} \end{array}\right\} \text{Ans.}$$

7.

$$\begin{array}{ll} (1) & x^2 - xy = 54 \\ (2) & xy - y^2 = 18, \text{ or } (x-y)y = 18 \end{array}$$

(1) — (2) gives $x^2 - 2xy + y^2 = 36$

Extracting root, $x - y = \pm 6$

Substitute in (2) factored gives

$$\pm 6y = 18$$

$$\left.\begin{array}{l} y = \pm 3 \\ x = \pm 6 \pm 3 = \pm 9 \end{array}\right\} \text{Ans.}$$

8.

$$x^2 - 9y^2 = 63$$
$$\frac{x}{2} = -2y$$
$$x = -4y$$
$$x^2 = 16y^2$$
$$16y^2 - 9y^2 = 63$$
$$7y^2 = 63$$
$$y^2 = 9$$
$$\left.\begin{aligned} y &= \pm 3 \\ x &= \mp 12 \end{aligned}\right\} \text{Ans.}$$

9.

$$\begin{aligned} 3x^2 + 4y^2 &= 76 \\ -11x^2 + 3y^2 &= 4 \\ \hline 9x^2 + 12y^2 &= 228 \\ -44x^2 + 12y^2 &= 16 \\ \hline 53x^2 &= 212 \\ x^2 &= 4 \end{aligned}$$
$$\left.\begin{aligned} x &= \pm 2 \\ y &= \pm 4 \end{aligned}\right\} \text{Ans.}$$

10.

$$3y^2 - x^2 = 39 \qquad (1)$$
$$x^2 + 4xy + 4y^2 = 256 \qquad (2)$$

From (2), $x + 2y = \pm 16$ (3)

First, $x = 16 - 2y$ (4)

Substituting in (1), we have

$$3y^2 - 256 + 64y - 4y^2 = 39$$

That is, $y^2 - 64y = -295$

Or, $y^2 - 64y + 1024 = 729$

Whence, $y - 32 = \pm 27$

$$y = 32 \pm 27$$
$$= 59, \text{ or } 5$$

Whence by (4), $x = -102$, or 6

Again, by (3), $x = -16 - 2y \qquad (5)$

Substituting in (1),

$$3y^2 - 256 - 64y - 4y^2 = 39$$

That is, $y^2 + 64y = -295$

Or, $y^2 + 64y + 1024 = 729$

Whence, $y + 32 = \pm 27$

$$y = -32 \pm 27$$

$$= -5, \text{ or } -59$$

Whence by (5), $x = -6$, or 102

Hence, $\left.\begin{array}{l} x = \pm 6, \text{ or } \pm 102 \\ y = \pm 5, \text{ or } \mp 59 \end{array}\right\}$ Ans.

11.

(1) $x^3 - y^3 = 19$

(2) $x^2y - xy^2 = 6$

Multiply (2) by -3 and add to (1),

$$x^3 - 3x^2y + 3xy^2 - y^3 = 1$$

$$(x - y)^3 = 1$$

$$x - y = 1$$

From (2), $xy(x - y) = 6$

$$\therefore xy = 6$$

By comparison, $x = 1 + y$

$$x = \frac{6}{y}$$

$$1 + y = \frac{6}{y}, \text{ or } y^2 + y = 6$$

Completing square and extracting root,

$$y^2 + y + \frac{1}{4} = \frac{24}{4} + \frac{1}{4}$$

$$y + \frac{1}{2} = \pm \frac{5}{2}$$

$\left.\begin{array}{l} y = 2, \text{ or } -3 \\ x = 3, \text{ or } -2 \end{array}\right\}$ Ans.

12.

(1) $x^2 + y^2 = 25$

(2) $xy = 12$

Multiply (2) by 2, add to (1), $x^2 + 2xy + y^2 = 49$

Subtract $2xy$ from (1), $x^2 - 2xy + y^2 = 1$

Extract square root in each case, $x + y = \pm 7$

$x - y = \pm 1$

Adding, $2x = \pm 7 \pm 1 = \pm 8$, or ± 6

Whence, $x = \pm 4$, or ± 3

Then by (2), $y = \pm 3$, or ± 4 } Ans.

(403.)

1.

By the conditions, $x^3 - y^3 = 117$ (1)

$$x = 2\tfrac{1}{2}y$$

$$x^3 = \frac{125}{8}y^3$$

Whence, from (1) we have $125y^3 - 8y^3 = 936$

$$117y^3 = 936$$

$$\left.\begin{aligned} y &= 2 \\ x = 2\tfrac{1}{2}y &= 5 \end{aligned}\right\} \text{Ans.}$$

2.

Let $x =$ one, $y =$ the other.

$$xy = 750$$

$$\frac{x}{y} = \frac{7}{2}$$

$$x = \frac{7}{2}y$$

$$xy = \frac{7}{2}y^2 = 750$$

$$y^2 = \frac{1500}{7}$$

$$\left.\begin{aligned} y &= 10\sqrt{\frac{15}{7}} \\ x &= 35\sqrt{\frac{15}{7}} \end{aligned}\right\} \text{Ans.}$$

3.

$$x^2 - y^2 = 25600$$
$$x : y = 5 : 3$$
$$5y = 3x$$
$$y = \frac{3}{5}x$$
$$y^2 = \frac{9}{25}x^2$$
$$x^2 - \frac{9}{25}x^2 = 25600$$
$$\frac{16}{25}x^2 = 25600$$
$$\frac{4}{5}x = 160$$
$$4x = 800$$
$$\left.\begin{aligned} x &= 200 \\ y &= 120 \end{aligned}\right\} \text{Ans.}$$

4.

$x =$ thickness.
$5x =$ height.
$40x =$ length.
$200x^3 = 5400$, cubical contents.
$x^3 = 27$

$$\left.\begin{aligned} x &= 3, \text{ thickness,} \\ 5x &= 15, \text{ height,} \\ 40x &= 120, \text{ length,} \end{aligned}\right\} \text{Ans.}$$

5.

$x =$ speed of coach.
$6 =$ distance.
$x - 5 =$ stüdent's rate.
$\frac{6}{x} =$ time of coach.
$\frac{6}{x} + \frac{50}{60} =$ time of student.
$\frac{6}{x-5} =$ " "

$$\frac{6}{x}+\frac{5}{6}=\frac{6}{x-5}$$
$$x^2-5x=36$$
$$x^2-5x+\frac{25}{4}=\frac{144}{4}+\frac{25}{4}$$
$$x-\frac{5}{2}=\pm\frac{13}{2}$$
$$x=9 \text{ miles per hour, Ans.}$$

6.

$$x = \text{cost.}$$
$$\text{Cost}+\text{gain}=\text{selling price.}$$
$$\frac{x}{100}=\text{rate per cent.}$$
$$\frac{x^2}{100}=\text{gain.}$$
$$x+\frac{x^2}{100}=\$39$$
$$x^2+100x=3900$$
$$x^2+100x+2500=6400$$
$$x+50=80$$
$$x=30, \text{ Ans.}$$

7.

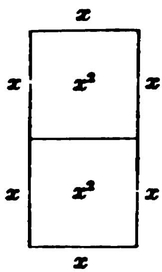
Let $x =$ number bought.

$$x-2= \text{ “ received.}$$
$$12 \text{ cents} = \text{sum expended.}$$
$$\frac{12}{x}=\text{market rate per peach.}$$
$$\frac{12}{x-2}=\text{rate paid.}$$

Also, $\frac{12}{x}+\frac{1}{15}=$ “ “

$$\frac{12}{x-2}=\frac{12}{x}+\frac{1}{15}$$
$$180x=180x-360+x^2-2x$$
$$x^2-2x=360$$
$$x^2-2x+1=361$$
$$x-1=\pm 19$$
$$\left.\begin{aligned} x&=20, \text{ number bought,}\\ x-2&=18, \text{ “ received,}\end{aligned}\right\}\text{ Ans.}$$

8.



$$2x^2(1.25) - 6x(.75) = \$22$$

$$x^2 - \frac{9}{5}x = \frac{44}{5}$$

$$x^2 - \frac{9}{5}x + \frac{81}{100} = \frac{44}{5} + \frac{81}{100} = \frac{961}{100}$$

$$x - \frac{9}{10} = \pm\frac{31}{10}$$

$$x = 4 \text{ ft.}$$

32 square feet in mirror.

\$40, cost of mirror.

\$40 — \$22 = \$18, cost of frame.

9.

$x =$ number of yds. bought.

$\$1.12 =$ price paid.

$\frac{112}{100x} =$ cost per yard in dollars.

$\frac{\frac{112}{100}}{x+1} =$ supposed cost.

$$\frac{112}{100x} - \frac{112}{100(x+1)} = \frac{2}{100}$$

$$112x + 112 - 112x = 2x^2 + 2x$$

$$x^2 + x = 56$$

$$x^2 + x + \frac{1}{4} = \frac{225}{4}$$

$$x = -\frac{1}{2} \pm \frac{15}{2} = 7$$

Ans. 7 yds. at 16 cents per yd.

11

$$x = \text{side of larger field.}$$
$$y = \quad \text{“} \quad \text{smaller “}$$

(1) $\qquad 4x + 4y = 200$

(2) $\qquad x^2 + y^2 = 1300$

(1) $\qquad x + y = 50$

Squaring (1), $\qquad x^2 + 2xy + y^2 = 2500$

Subtracting (2), $\qquad 2xy = 1200 \qquad$ (3)

Subtracting (2) from (3), $\qquad x^2 - 2xy + y^2 = 100$

Whence, $\qquad x - y = 10 \qquad$ (4)

Adding (1) and (4), $\qquad 2x = 50 + 10 = 60$

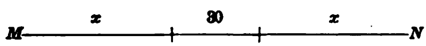
Whence, $\qquad x = 30$

Then by (1), $\qquad y = 20$

Therefore, cost of larger field $= 900 \times 2\frac{1}{4} = \$2025$,

And cost of smaller $= 400 \times 2\frac{1}{4} = \$900$, Ans.

12.



$$2x + 30 = \text{whole distance.}$$

Since A travels B's distance, x, in 4 days, his rate is $\frac{x}{4}$, and for like reason B's rate $= \dfrac{x + 30}{9}$.

Since the law of locomotion is

$$\text{Distance} = \text{Time} \times \text{Rate},$$

and they travel *the same time* before meeting,

$$\text{A's time} = \text{B's time.}$$

Hence, $\qquad \dfrac{4(x + 30)}{x} = \dfrac{9x}{x + 30}$

$$4(x + 30)^2 = 9x^2$$

Extracting root, $\qquad 2x + 60 = 3x$

$$60 = x$$

$$2x + 30 = 150 \text{ miles.}$$

$$\frac{60}{4} = 15 \text{ miles, A's rate.}$$

$$\frac{60 + 30}{9} = 10 \text{ miles, B's rate.}$$

(104.)

1.

(1) $$l = a + (n - 1)\, d$$

(2) $$S = \frac{n}{2}(a + l)$$

Given $n = 7$, $a = 5$, $d = 3$, to find l.

By (1), $l = 5 + 18 = 23$, Ans.

2.

Given $a = \frac{1}{3}$, $d = -\frac{1}{12}$, $S = -\frac{3}{2}$,

to find number of terms.

By (1), $$l = \frac{1}{3} - \frac{n-1}{12} \qquad (3)$$

By (2), $$-\frac{3}{2} = \frac{n}{2}\left(\frac{1}{3} + l\right) \qquad (4)$$

Substituting from (3) in (4), we have

$$-3 = n\left[\frac{2}{3} - \frac{n-1}{12}\right] = \frac{3n}{4} - \frac{n^2}{12}$$

Then, $$n^2 - 9n = 36$$

$$n^2 - 9n + \frac{81}{4} = 36 + \frac{81}{4} = \frac{225}{4}$$

$$n - \frac{9}{2} = \pm \frac{15}{2}$$

$$n = \frac{9}{2} \pm \frac{15}{2} = 12, \text{ Ans.}$$

3.

Given $a = 3$, $n = 20$, $S = 440$.

(1) $$l = 3 + 19\, d$$

(2) $$440 = 60 + 190\, d$$

$$d = 2, \text{ Ans.}$$

4.

(1) $$\left.\begin{aligned} l &= 1 + (15 - 1)\,5 = 71, \text{ last term,} \\ S &= \frac{15}{2}(1 + 71) = 540, \text{ sum of series,} \end{aligned}\right\} \text{Ans.}$$

5.

Given $d=5$, $n=6$, $S=321$,
to find first and last term.

Let $x=$ 1st term, $y=$ last term.

We then have from the formulas,

$$y=x+25$$
$$321=3(x+y)$$

Solution.

$$321=3x+3x+75$$
$$246=6x$$
$$\left.\begin{aligned}41&=x\\ y&=41+25=66\end{aligned}\right\}\text{ Ans.}$$

6.

Given $a=3$, $l=42\tfrac{2}{3}$, $d=2\tfrac{1}{3}$,
to find S and n.

(1) $$42\tfrac{2}{3}=3+\frac{7}{3}n-\frac{7}{3}$$

$n=18$, number of terms.

(2) $$S=\frac{18}{2}(3+42\tfrac{2}{3})$$

$=9\times 45\tfrac{2}{3}=411$, sum of the terms, Ans.

7.

In the following examples use

(1) $$l=ar^{n-1}$$

(2) $$S=a\left(\frac{r^n-1}{r-1}\right)$$

(1) $$128=a\,2^{6-1}$$

Hence, $$32a=128$$

$a=4$, Ans.

8.

To find r when $a = 2, \quad l = 4374, \quad n = 8.$

$$4374 = 2\,r^{8-1}$$
$$r^7 = 2187$$
$$r = 3, \text{ Ans.}$$

9.

Given $a = \frac{1}{2}, \quad l = 128, \quad n = 5.$

$$128 = \left(\frac{1}{2}\right) r^4$$
$$r^4 = 256$$
$$r^2 = 16$$
$$r = \pm 4$$

$$\left.\begin{array}{lllll} \frac{1}{2}, & 2, & 8, & 32, & 128 \\ \text{or, } \frac{1}{2}, & -2, & 8, & -32, & 128 \end{array}\right\} \text{Ans.}$$

10.

Find the value of .1212 to infinity.

This is a circulating decimal, and may be expressed

$$\frac{12}{100}, \frac{12}{10000} \ldots \text{ to infinity.}$$
$$a = \frac{12}{100}, \quad r = \frac{1}{100}$$
$$S = \frac{a}{1 - r} \quad \text{(Art. 345)}$$
$$S = \frac{12}{100} \div \left(1 - \frac{1}{100}\right)$$
$$\frac{12}{100} \times \frac{100}{99} = \frac{12}{99} = \frac{4}{33}, \text{ Ans.}$$

MISCELLANEOUS EXERCISES.

(405.)

1.

See page 24.

2.

$$\frac{a+b}{b^2} - \frac{a-b}{ab} = \frac{a(a+b) - b(a-b)}{ab^2} = \frac{a^2+b^2}{ab^2}$$

$$\frac{a^2+b^2}{ab^2} \times \frac{1}{ab} = \frac{a^2+b^2}{a^2b^3}$$

$$\frac{a^2+b^2}{a^2b^3} \div \frac{a-b}{ab} = \frac{a^2+b^2}{a^2b^3} \times \frac{ab}{a-b} = \frac{a^2+b^2}{ab^2(a-b)}, \text{ Ans.}$$

3.

$$\begin{array}{r} x^2 - 12x + 35\,)\; x^3 - 2x^2 - 19x + 20\;(\,x + 10 \\ \underline{x^3 - 12x^2 + 35x\phantom{{}+20}} \\ 10x^2 - 54x + 20 \\ \underline{10x^2 - 120x + 350} \\ 66x - 330 \end{array}$$

$$\begin{array}{r} x - 5\,)\; x^2 - 12x + 35\;(\,x - 7 \\ \underline{x^2 - 5x\phantom{{}+35}} \\ -7x + 35 \\ \underline{-7x + 35} \end{array} \qquad \text{G. C. D.} = x - 5, \text{ Ans.}$$

$$\begin{array}{l} x^3 - 2x^2 - 19x + 20 \\ \underline{x - 7} \\ x^4 - 2x^3 - 19x^2 + 20x \\ \underline{ - 7x^3 + 14x^2 + 133x - 140} \\ x^4 - 9x^3 - 5x^2 + 153x - 140 = \text{L. C. M., Ans.} \end{array}$$

4.

$$3a - 9 - \frac{3a-30}{a+3} = \frac{3a^2 - 9a + 9a - 27 - 3a + 30}{a+3}$$

$$= \frac{3a^2 - 3a + 3}{a+3}, \text{ Ans.}$$

5.

See page 103.

6.

Given $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x+19}{2}$

Clearing of fractions, $3x - 9 + 2x = 120 - 3x - 57$

Transposing and uniting terms, $8x = 72$

Whence, $x = 9$, Ans.

7.

Let $x =$ their income.

Then $\frac{2x}{7} =$ the debt which A contracts in two years,

and $\frac{2x}{5} =$ the amount which B saves in two years.

By the conditions, $\frac{2x}{5} - \frac{2x}{7} = 32$

Dividing by 2, $\frac{x}{5} - \frac{x}{7} = 16$

Clearing of fractions, $7x - 5x = 560$

That is, $2x = 560$

Whence, $x = 280$, Ans.

8.

See page 132.

9.

Let $x =$ the number of days he was idle.

Then $a - x =$ the number of days he worked.

By the conditions, $b(a - x) - cx = d$

That is, $ab - bx - cx = d$

Transposing, $ab - d = bx + cx$

That is, $x(b + c) = ab - d$

Whence, $x = \frac{ab - d}{b + c}$, Ans.

10.

Let $x =$ the number of persons,

and $y =$ the sum of money divided.

Then $\frac{y}{x} =$ what each received.

By the conditions, $$\frac{y}{x + 3} = \frac{y}{x} - 1 \quad (1)$$

$$\frac{y}{x - 2} = \frac{y}{x} + 1 \quad (2)$$

Adding (1) and (2), $$\frac{y}{x + 3} + \frac{y}{x - 2} = \frac{2y}{x} \quad (3)$$

Dividing (3) by y, $$\frac{1}{x + 3} + \frac{1}{x - 2} = \frac{2}{x}$$

Clearing of fractions,

$$x^2 - 2x + x^2 + 3x = 2x^2 + 2x - 12$$

Transposing and uniting, $-x = -12$

Whence, $x = 12$

Substituting in (1), $\frac{y}{15} = \frac{y}{12} - 1$

That is, $-\frac{y}{60} = -1$

Whence, $y = 60$

Ans. $\begin{cases} \text{Number of persons} = 12. \\ \text{Each received } \$5. \end{cases}$

(406.)

1.

$$(x-a)(x+a)(x^2+a^2)=(x^2-a^2)(x^2+a^2)$$
$$=x^4-a^4, \text{ Ans.}$$

2.

$$\begin{array}{r|l}
x-3\,) & x^2-7x+12\,(\,x-4, \text{ Ans.} \\
 & \underline{x^2-3x} \\
 & \quad -4x+12 \\
 & \quad \underline{-4x+12}
\end{array}$$

3.

$$\begin{array}{r|l}
x^2+4x+8\,) & x^4+64\,(\,x^2-4x+8, \text{ Ans.} \\
 & \underline{x^4+4x^3+\ \ 8x^2} \\
 & \quad -4x^3-\ \ 8x^2+64 \\
 & \quad \underline{-4x^3-16x^2-32x} \\
 & \qquad\qquad 8x^2+32x+64 \\
 & \qquad\qquad \underline{8x^2+32x+64}
\end{array}$$

4.

$$\frac{4(a+b)^2}{5(a^2-b^2)}=\frac{4(a+b)(a+b)}{5(a+b)(a-b)}=\frac{4(a+b)}{5(a-b)}, \text{ Ans.}$$

5.

$$\frac{a}{a-x}+\frac{3a}{a+x}+\frac{2ax}{a^2-x^2}=\frac{a(a+x)+3a(a-x)+2ax}{a^2-x^2}$$
$$=\frac{a^2+ax+3a^2-3ax+2ax}{a^2-x^2}$$
$$=\frac{4a^2}{a^2-x^2}, \text{ Ans.}$$

6.

Given $\frac{x-5}{7}=\frac{x+5}{9}$

Clearing of fractions, $9x-45=7x+35$

That is, $2x=80$

Whence, $x=40$, Ans.

7.

Let $x =$ number of lbs. of the latter sort.

Then $50 + x =$ " " in the mixture.

By the conditions, $50 \times 60 + 25x = 50(50 + x)$

That is, $3000 + 25x = 2500 + 50x$

Transposing and uniting, $500 = 25x$

Whence, $x = 20$, Ans.

8.

Given $$\frac{x+y}{3} + \frac{y-x}{2} = 9 \quad (1)$$

$$\frac{x}{2} + \frac{x+y}{9} = 5 \quad (2)$$

From (1), $2x + 2y + 3y - 3x = 54$

That is, $5y - x = 54$ (3)

From (2), $9x + 2x + 2y = 90$

That is, $11x + 2y = 90$ (4)

Multiplying (3) by 11, $55y - 11x = 594$ (5)

Adding (4) and (5), $57y = 684$

Whence, $y = 12$

Substituting in (3), $60 - x = 54$

Whence, $x = 6$

Ans. $\begin{cases} x = 6. \\ y = 12. \end{cases}$

9.

Let $x =$ the length of the floor,

and $y =$ its breadth.

By the conditions, $(x + 3)(y + 2) = xy + 64$ (1)

$(x + 2)(y + 3) = xy + 68$ (2)

From (1), $xy + 2x + 3y + 6 = xy + 64$

That is, $2x + 3y = 58$ (3)

From (2), $xy + 3x + 2y + 6 = xy + 68$

That is, $3x + 2y = 62$ (4)

Multiplying (3) by 2, $4x + 6y = 116$ (5)

Multiplying (4) by 3, $9x + 6y = 186$ (6)

Subtracting (5) from (6), $5x = 70$

Whence, $x = 14$

Substituting in (4), $42 + 2y = 62$

That is, $2y = 20$

Whence, $y = 10$

Ans. $\begin{cases} \text{Length} = 14 \text{ ft.} \\ \text{Breadth} = 10 \text{ ft.} \end{cases}$

10.

Given $\dfrac{2x + 11}{x} = 5 - \dfrac{x - 5}{3}$

Clearing of fractions, $6x + 33 = 15x - x^2 + 5x$

Transposing and uniting, $x^2 - 14x = -33$

Completing the square, $x^2 - 14x + 49 = 49 - 33 = 16$

Extracting the square root, $x - 7 = \pm 4$

That is, $x = 7 \pm 4$

Whence, $x = 11$, or 3, Ans.

(407.)

1.

$$\frac{x+y}{x-y} - \frac{x-y}{x+y} = \frac{(x+y)^2 - (x-y)^2}{x^2 - y^2}$$

$$= \frac{x^2 + 2xy + y^2 - x^2 + 2xy - y^2}{x^2 - y^2}$$

$$= \frac{4xy}{x^2 - y^2}, \text{ Ans.}$$

2.

$$\frac{\frac{a+b}{2}}{\frac{a-b}{3}}-\frac{\frac{a-b}{3}}{\frac{a+b}{2}}=\frac{3(a+b)}{2(a-b)}-\frac{2(a-b)}{3(a+b)}$$

$$=\frac{9(a+b)^2-4(a-b)^2}{6(a^2-b^2)}$$

$$=\frac{9a^2+18ab+9b^2-4a^2+8ab-4b^2}{6(a^2-b^2)}$$

$$=\frac{5a^2+26ab+5b^2}{6(a^2-b^2)},\text{ Ans.}$$

3.

$$\frac{x}{7\frac{1}{2}}=\frac{x}{\frac{15}{2}}=\frac{2x}{15}=\frac{2x^2}{15x}$$

$$\frac{5\frac{1}{3}}{x}=\frac{\frac{16}{3}}{x}=\frac{16}{3x}=\frac{80}{15x}$$

4.

Given

$$\frac{6x+18}{13}-4\tfrac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}-\frac{21-2x}{18}$$

Clearing of fractions,

$$216x+648-2262-143+39x$$
$$=2340x-22464-507+39x-546+52x$$

Transposing,

$$216x+39x-2340x-39x-52x$$
$$=-648+2262+143-22464-507-546$$

Uniting terms, $\qquad -2176x=-21760$

Whence, $\qquad x=10$, Ans.

5.

Given $5x - \frac{4x-a}{b} + \frac{2x+2a}{4} = m + n - \frac{2x+2a}{c}$

Clearing of fractions, $20bcx - 16cx + 4ac + 2bcx + 2abc$
$= 4mbc + 4nbc - 8bx - 8ab$

Transposing and uniting, $22bcx - 16cx + 8bx$
$= 4bcm + 4bcn - 8ab - 4ac - 2abc$

Dividing by 2, $(11bc - 8c + 4b)x$
$= 2bcm + 2bcn - 4ab - 2ac - abc$

Whence, $x = \frac{2bcm + 2bcn - 4ab - 2ac - abc}{11bc - 8c + 4b}$, Ans.

6.

Let $x =$ amount given to a man,

and $y =$ " " " woman.

By the conditions, $5x + 7y = 46$ (1)

$x + y = 8$ (2)

Multiplying (2) by 5, $5x + 5y = 40$ (3)

Subtracting (3) from (1), $2y = 6$

Whence, $y = 3$

Substituting in (2), $x + 3 = 8$

Whence, $x = 5$

Ans. { To the men, \$25.
" women, \$21.

7.

Let $x =$ the number.

By the conditions, $3x + 12 - 54 = 144 - 3x$

Transposing and uniting, $6x = 186$

Whence, $x = 31$, Ans.

8.

Let $x =$ amount of his money.

Then, $\frac{3x}{8}$ is lent at 5 per cent,

and $\frac{5x}{8}$ at 6 per cent.

By the conditions, $\frac{3x}{8} \times \frac{5}{100} + \frac{5x}{8} \times \frac{6}{100} = 144$

Reducing, $\frac{3x}{160} + \frac{3x}{80} = 144$

Clearing of fractions, $3x + 6x = 23040$

That is, $9x = 23040$

Whence, $x = 2560$

Then, $\frac{3x}{8} = 960$

and $\frac{5x}{8} = 1600$

Ans. $\begin{cases} \$960, \text{ at 5 per cent.} \\ \$1600, \text{ at 6 per cent.} \end{cases}$

9.

See page 132.

10.

Given
$$4x + y = 34 \qquad (1)$$
$$4y + x = 16 \qquad (2)$$

1st. *By Substitution.*

From (1), $y = 34 - 4x$

Substituting in (2), $136 - 16x + x = 16$

Whence, $-15x = -120$

Or, $x = 8$

Substituting in (1), $32 + y = 34$

Whence, $y = 2$

2d. *By Comparison.*

From (1), $y = 34 - 4x$ (3)

From (2), $4y = 16 - x$

Or, $y = \frac{16 - x}{4}$ (4)

Equating (3) and (4), $34 - 4x = \frac{16 - x}{4}$

Clearing of fractions, $136 - 16x = 16 - x$

Transposing and uniting, $-15x = -120$

Whence, $x = 8$

Substituting in (3), $y = 34 - 32 = 2$

Ans. $\begin{cases} x = 8. \\ y = 2. \end{cases}$

(408.)

1.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}, \text{ Ans.}$$

As to the reason, see Art. 140.

2.

To reduce a fraction to its simplest form.—If the numerator and denominator be divided by their greatest common divisor, the fraction will be reduced to its simplest form.

3.

In the addition of fractions.—If the least common multiple of the denominators be found, it will be the least common denominator of the fractions.

4.

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad$$
$$+2bc+2bd+2cd, \text{ Ans.}$$

5.

$$\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}-\frac{4ab}{4b^2-x^2}$$
$$=\frac{(x+2a)(2b+x)+(x-2a)(2b-x)-4ab}{4b^2-x^2}$$
$$=\frac{2bx+x^2+4ab+2ax+2bx-x^2-4ab+2ax-4ab}{4b^2-x^2}$$
$$=\frac{4ax+4bx-4ab}{4b^2-x^2}=\frac{4x(a+b)-4ab}{4b^2-x^2}$$

Putting in this, $x=\frac{ab}{a+b}$, we have,

$$\frac{4ab-4ab}{4b^2-\frac{a^2b^2}{(a+b)^2}}=0, \text{ Ans.}$$

6.

Given $\quad \frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}$

Transposing, $\quad \frac{6x+7}{15}-\frac{2x+1}{5}=\frac{2x-2}{7x-6}$

Adding the fractions in the first member,

$$\frac{6x+7-6x-3}{15}=\frac{2x-2}{7x-6}$$

Reducing, $\quad \frac{4}{15}=\frac{2x-2}{7x-6}$

Clearing of fractions, $\quad 28x-24=30x-30$

Transposing and uniting, $\quad -2x=-6$

Whence, $\quad x=3$, Ans.

7.

Let $x =$ number he buys.

Then, $\frac{2x}{5} =$ amount he pays for them.

By the conditions, $\frac{x}{2} \times \frac{1}{2} + \frac{x}{2} \times \frac{1}{3} = \frac{2x}{5} + 1$

Reducing, $\frac{x}{4} + \frac{x}{6} = \frac{2x}{5} + 1$

Clearing of fractions, $15x + 10x = 24x + 60$

Transposing and uniting, $x = 60$, Ans.

8.

Given $x^3 + y^3 = 91$ (1)

$x + y = 7$ (2)

Dividing (1) by (2), $x^2 - xy + y^2 = 13$ (3)

Squaring (2), $x^2 + 2xy + y^2 = 49$ (4)

Subtracting (3) from (4), $3xy = 36$

Whence, $xy = 12$ (5)

Subtracting (5) from (3),

$$x^2 - 2xy + y^2 = 1$$

Extracting the square root, $x - y = \pm 1$ (6)

Adding (2) and (6), $2x = 7 \pm 1 = 8$, or 6

Whence, $x = 4$, or 3

Subtracting (6) from (2), $2y = 7 \mp 1 = 6$, or 8

Whence, $y = 3$, or 4

$$\text{Ans.} \begin{cases} x = 4, \text{ or } 3. \\ y = 3, \text{ or } 4. \end{cases}$$

9.

$$\begin{aligned}
\because \sqrt[3]{128} &= \sqrt[3]{64 \times 2} = \sqrt[3]{64} \times \sqrt[3]{2} = 4\sqrt[3]{2}\\
\sqrt[3]{686} &= \sqrt[3]{343 \times 2} = \sqrt[3]{343} \times \sqrt[3]{2} = 7\sqrt[3]{2}\\
\sqrt[3]{16} &= \sqrt[3]{8 \times 2} = \sqrt[3]{8} \times \sqrt[3]{2} = 2\sqrt[3]{2}\\
7\sqrt[3]{54} &= 7\sqrt[3]{27 \times 2} = 7\sqrt[3]{27} \times \sqrt[3]{2} = 21\sqrt[3]{2}\\
3\sqrt[3]{16} &= 3\sqrt[3]{8 \times 2} = 3\sqrt[3]{8} \times \sqrt[3]{2} = 6\sqrt[3]{2}\\
\sqrt[3]{432} &= \sqrt[3]{216 \times 2} = \sqrt[3]{216} \times \sqrt[3]{2} = 6\sqrt[3]{2}
\end{aligned}$$

Adding, we have,

$$(4 + 7 + 2 + 21 + 6 + 6)\sqrt[3]{2} = 46\sqrt[3]{2}, \text{ Ans.}$$

10.

$$\frac{1}{\sqrt{5} - \sqrt{2}} = \frac{\sqrt{5} + \sqrt{2}}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})} = \frac{\sqrt{5} + \sqrt{2}}{5 - 2} = \frac{\sqrt{5} + \sqrt{2}}{3}$$

$$\begin{array}{r|l}
 & \dot{5}.\dot{0}0\dot{0}0\dot{0}0 \;\;|\; 2.236+ \\
 & 4 \\
\hline
42 & 100 \\
 & 84 \\
\hline
443 & 1600 \\
 & 1329 \\
\hline
4466 & 27100 \\
 & 26796 \\
\hline
\end{array}
\qquad
\begin{array}{r|l}
 & \dot{2}.\dot{0}0\dot{0}0\dot{0}0 \;\;|\; 1.414+ \\
 & 1 \\
\hline
24 & 100 \\
 & 96 \\
\hline
281 & 400 \\
 & 281 \\
\hline
2824 & 11900 \\
 & 11296 \\
\hline
\end{array}$$

Then, our result $= \dfrac{2.236 + 1.414}{3} = \dfrac{3.650}{3} = 1.217+$, Ans.

(409.)

1.

$$\begin{array}{r}
4a^2x - 2b^4y + a^2y^2 \\
-3a^2x - 4b^4y + 5a^2y^2 - 3(x - y)\\
\hline
7a^2x + 2b^4y - 4a^2y^2 + 3(x - y), \text{ Ans.}
\end{array}$$

2.

$$x^3 - x = x(x^2 - 1) = x(x+1)(x-1), \text{ Ans.}$$

3.

$$\begin{aligned} x^3 - 1 &= (x-1)(x^2+x+1) \\ a x^4 - a x &= a x(x^3 - 1) = a x(x-1)(x^2+x+1) \\ x+1 &= x+1 \\ \hline \therefore \text{L. C. M.} &= a x(x+1)(x-1)(x^2+x+1) \\ &= (a x^2 + a x)(x^3 - 1) \\ &= a x^5 + a x^4 - a x^2 - a x, \text{ Ans.} \end{aligned}$$

4.

$$\frac{9(a^2-b^2)c^{4-n}x^{-2}y}{3(a^2-b^2)c^{-2}xy^2} = 3c^{6-n}x^{-3}y^{-1}, \text{ Ans.}$$

5.

Given $\quad a x - x = 2 + c$

Factoring, $\quad x(a-1) = 2 + c$

Whence, $\quad x = \dfrac{2+c}{a-1}$, Ans.

Verification.

$$a\frac{2+c}{a-1} - \frac{2+c}{a-1} = (a-1)\frac{2+c}{a-1} = 2 + c$$

6.

Given $\quad x + y + z = 26$

$x - y = 4 \qquad (1)$

$x - z = 6 \qquad (2)$

Adding the three equations, $\quad 3x = 36$

Whence, $\quad x = 12 \qquad (3)$

Substituting from (3) in (1), $12 - y = 4$

Whence, $\quad y = 8$

Substituting from (3) in (2), $12 - z = 6$

Whence, $\quad z = 6$

$$\text{Ans.} \begin{cases} x = 12. \\ y = 8. \\ z = 6. \end{cases}$$

7.

$$\frac{a}{2a-2b}+\frac{b}{2b-2a}=\frac{a}{2a-2b}-\frac{b}{2a-2b}$$

$$=\frac{a-b}{2(a-b)}=\frac{1}{2}, \text{ Ans.}$$

8.

Let $\qquad x =$ number of hours A requires,

and $\qquad y =$ " " B "

By the conditions, $\qquad x = y - 9 \qquad (1)$

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{20} \qquad (2)$$

From (2), $\qquad 20(x+y)=xy \qquad (3)$

Substituting from (1) in (3),

$$20(y+y-9)=y(y-9)$$

Reducing, $\qquad y^2-49y=-180$

Completing the square,

$$y^2-49y+\frac{2401}{4}=-180+\frac{2401}{4}=\frac{1681}{4}$$

Extracting the square root,

$$y-\frac{49}{2}=\pm\frac{41}{2}$$

That is, $\qquad y=\frac{49}{2}\pm\frac{41}{2}=45, \text{ or } 4$

Substituting in (1), $\qquad x=45-9, \text{ or } 4-9$

That is, $\qquad x=36, \text{ or } -5$

Ans. $\begin{cases} \text{A takes 36 hours.} \\ \text{B } \quad \text{"} \quad \text{45} \quad \text{"} \end{cases}$

9.

See pp. 12 and 104.

10.

Let $x =$ rate at which the river flows.

Then, $12 + x =$ " of the crew down stream,

and $12 - x =$ " " " up stream.

By the conditions, $7(12 - x) = 5(12 + x)$

Reducing, $84 - 7x = 60 + 5x$

Transp. and uniting, $-12x = -24$

Whence, $x = 2$, Ans.

(410.)

1.

$$\begin{aligned} a^3 - 1 &= (a - 1)(a^2 + a + 1) \\ a^2 - 1 &= (a + 1)(a - 1) \\ a + 1 &= a + 1 \\ \hline \text{L. C. M.} &= (a + 1)(a - 1)(a^2 + a + 1) \\ &= (a + 1)(a^3 - 1) \\ &= a^4 + a^3 - a - 1, \text{ Ans.} \end{aligned}$$

2.

$$\frac{1 + a}{a} - \frac{a}{-a} = \frac{1 + a}{a} + 1; \text{ difference} = \frac{2a + 1}{a}$$

$$\frac{a}{-a} \div \frac{1 + a}{a} = -1 \times \frac{a}{1 + a} = -\frac{a}{1 + a}$$

$$\text{Quotient} = -\frac{a}{1 + a}, \text{ Ans.}$$

3.

$$\frac{a + \frac{b}{c}}{x - \frac{m}{n}} = \frac{\frac{ac + b}{c}}{\frac{nx - m}{n}} = \frac{ac + b}{c} \times \frac{n}{nx - m} = \frac{nac + bn}{cnx - cm}, \text{ Ans.}$$

4.

See Art. 71.

5.

See Art. 71.

6.

Let $x =$ the greater number,
and $y =$ the less.

By the conditions, $x - y = 2$ (1)

$x^2 + y^2 = 164$ (2)

Squaring (1), $x^2 - 2xy + y^2 = 4$ (3)

Subtracting (3) from (2), $2xy = 160$ (4)

Adding (2) and (4), $x^2 + 2xy + y^2 = 324$ (5)

Extracting the square root, $x + y = \pm 18$ (6)

Adding (1) and (6), $2x = 2 \pm 18 = 20$, or -16

Whence, $x = 10$, or -8

Subtracting (1) from (6), $2y = -2 \pm 18 = 16$, or -20

Whence, $y = 8$, or -10

Ans. $\begin{cases} \text{The greater number} = 10. \\ \text{The less number} = 8. \end{cases}$

7.

Given $x^2 + y^2 = 41$ (1)

$xy = 20$ (2)

Multiplying (2) by 2, $2xy = 40$ (3)

Adding (1) and (3),

$$x^2 + 2xy + y^2 = 81$$

Extracting the square root,

$$x + y = \pm 9 \qquad (4)$$

Subtracting (3) from (1),

$$x^2 - 2xy + y^2 = 1$$

Extracting the square root,

$$x - y = \pm 1 \qquad (5)$$

Adding (4) and (5), $2x = \pm 9 \pm 1 = 10, 8, -8$, or -10

Whence, $x = 5, 4, -4$, or -5

Subtracting (5) from (4),

$$2y = \pm 9 \mp 1 = 8, 10, -10, \text{ or } -8$$

Whence, $y = 4, 5, -5$, or -4

Ans. $\begin{cases} x = 5, 4, -4, \text{ or } -5. \\ y = 4, 5, -5, \text{ or } -4. \end{cases}$

5.

Given $5x - \frac{4x-a}{b} + \frac{2x+2a}{4} = m + n - \frac{2x+2a}{c}$

Clearing of fractions, $20bcx - 16cx + 4ac + 2bcx + 2abc$
$= 4mbc + 4nbc - 8bx - 8ab$

Transposing and uniting, $22bcx - 16cx + 8bx$
$= 4bcm + 4bcn - 8ab - 4ac - 2abc$

Dividing by 2, $(11bc - 8c + 4b)x$
$= 2bcm + 2bcn - 4ab - 2ac - abc$

Whence, $x = \frac{2bcm + 2bcn - 4ab - 2ac - abc}{11bc - 8c + 4b}$, Ans.

6.

Let $x =$ amount given to a man,
and $y =$ " " " woman.

By the conditions, $5x + 7y = 46$ (1)
$x + y = 8$ (2)
Multiplying (2) by 5, $5x + 5y = 40$ (3)
Subtracting (3) from (1), $2y = 6$
Whence, $y = 3$
Substituting in (2), $x + 3 = 8$
Whence, $x = 5$

Ans. $\begin{cases} \text{To the men, \$25.} \\ \text{" women, \$21.} \end{cases}$

7.

Let $x =$ tne number.
By the conditions, $3x + 12 - 54 = 144 - 3x$
Transposing and uniting, $6x = 186$
Whence, $x = 31$, Ans.

8.

Let $x =$ amount of his money.

Then, $\frac{3x}{8}$ is lent at 5 per cent,

and $\frac{5x}{8}$ at 6 per cent.

By the conditions, $\frac{3x}{8} \times \frac{5}{100} + \frac{5x}{8} \times \frac{6}{100} = 144$

Reducing, $\frac{3x}{160} + \frac{3x}{80} = 144$

Clearing of fractions, $3x + 6x = 23040$

That is, $9x = 23040$

Whence, $x = 2560$

Then, $\frac{3x}{8} = 960$

and $\frac{5x}{8} = 1600$

Ans. $\begin{cases} \$960, \text{ at 5 per cent.} \\ \$1600, \text{ at 6 per cent.} \end{cases}$

9.

See page 132.

10.

Given
$$4x + y = 34 \qquad (1)$$
$$4y + x = 16 \qquad (2)$$

1st. *By Substitution.*

From (1), $y = 34 - 4x$

Substituting in (2), $136 - 16x + x = 16$

Whence, $-15x = -120$

Or, $x = 8$

Substituting in (1), $32 + y = 34$

Whence, $y = 2$

5.

$$
\begin{array}{l|l}
 & 9x^6 - 24x^4y^2 - 12x^3y^3 + 16x^2y^4 + 16xy^5 + 4y^6 \quad \big|\, 3x^3 - 4xy^2 - 2y^3, \text{ Ans.} \\
 & 9x^6 \\
\hline
6x^3 - 4xy^2 & -24x^4y^2 \\
\quad -4xy^2 & -24x^4y^2 + 16x^2y^4 \\
\hline
6x^3 - 8xy^2 - 2y^3 & -12x^3y^3 + 16xy^5 + 4y^6 \\
\quad -2y^3 & -12x^3y^3 + 16xy^5 + 4y^6 \\
\hline
\end{array}
$$

6.

Given

$$y - a = 2(x - b) \qquad (1)$$

$$y - b = 2(x - a) \qquad (2)$$

Subtracting (2) from (1), we get $b - a = 2(a - b)$, which is impossible. Therefore the given equations are impossible.

7.

$$
\begin{aligned}
\sqrt[3]{4a} \times \sqrt{6x} &= \sqrt[6]{16a^2} \times \sqrt[6]{216x^3} \\
&= \sqrt[6]{3456\, a^2 x^3} \\
&= \sqrt[6]{64 \times 54\, a^2 x^3} \\
&= 2\sqrt[6]{54\, a^2 x^3}, \text{ Ans.}
\end{aligned}
$$

8.

Let $x =$ number of quarts of the first quality,

and $100 - x$ " " " " second.

By the conditions, $90x + 36(100 - x) = 5000$

Reducing, $54x = 1400$

Whence, $x = \frac{1400}{54} = \frac{700}{27} = 25\frac{25}{27}$

Therefore, $100 - x = 74\frac{2}{27}$

Ans. $\begin{cases} 25\frac{25}{27} \text{ quarts of the first quality,} \\ 74\frac{2}{27} \quad \text{"} \quad \text{"} \quad \text{second.} \end{cases}$

9.

$$\begin{array}{r|l} x^{\frac{1}{3}} - 2 \,)\; x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 & (\, x^{\frac{1}{3}} + 3, \text{ Ans.} \\ \underline{x^{\frac{2}{3}} - 2\,x^{\frac{1}{3}}} & \\ 3\,x^{\frac{1}{3}} - 6 & \\ \underline{3\,x^{\frac{1}{3}} - 6} & \end{array}$$

10.

$$\left[\left(\frac{-a^2 b^3}{c^3 d}\right)^2\right]^3 = \left(\frac{-a^2 b^3}{c^3 d}\right)^6 = \frac{a^{12} b^{18}}{c^{18} d^6}, \text{ Ans.}$$

(412.)

1.

$$1 - \frac{a}{a - x} - \frac{x^2}{a^2 - x^2} = \frac{a^2 - x^2 - a\,(a + x) - x^2}{a^2 - x^2}$$

$$= \frac{a^2 - x^2 - a^2 - a\,x - x^2}{a^2 - x^2}$$

$$= -\frac{2\,x^2 + a\,x}{a^2 - x^2}, \text{ Ans.}$$

2.

$$x^6 - 64\,a^6 = (x^3 + 8\,a^3)\;(x^3 - 8\,a^3)$$

$$= (x + 2\,a)\;(x^2 - 2\,a\,x + 4\,a^2)\;(x - 2\,a)$$

$$(x^2 + 2\,a\,x + 4\,a^2), \text{ Ans.}$$

3.

Given $$\frac{x}{a} + \frac{x}{b - a} = \frac{a}{a + b}$$

Clearing of fractions,

$$x\,(b^2 - a^2) + x\,a\,(a + b) = a^2\,(b - a)$$

Factoring, $x\,(b^2 - a^2 + a^2 + a\,b) = a^2\,(b - a)$

Whence, $$x = \frac{a^2\,(b - a)}{b^2 + a\,b}, \text{ Ans.}$$

4.

Given	$7(x+y)+3(x-y)=80$	(1)
	$7(x+y)-3(x-y)=32$	(2)
From (1),	$10x+4y=80$	(3)
From (2),	$4x+10y=32$	(4)
Multiplying (3) by $\frac{5}{2}$,	$25x+10y=200$	(5)
Subtracting (4) from (5),	$21x=168$	
Whence,	$x=8$	
Substituting in (3),	$80+4y=80$	
Whence,	$y=0$	

$$\text{Ans. } \begin{cases} x=8. \\ y=0. \end{cases}$$

5.

$$\begin{array}{r}
x^3+3x^2+4x+12\;)\;x^3+4x^2+4x+3\;(1 \\
\underline{x^3+3x^2+4x+12} \\
x^2\qquad\;\, -\;\;9
\end{array}$$

$$\begin{array}{r}
x^2-9\;)\;x^3+3x^2+4x+12\;(x+3 \\
\underline{x^3\qquad\;\;\, -9x\qquad} \\
3x^2+13x+12 \\
\underline{3x^2\qquad\;\; -27} \\
13x+39
\end{array}$$

$$\begin{array}{r}
x+3\;)\;x^2-9\;(x-3 \\
\underline{x^2+3x} \\
-3x-9 \\
\underline{-3x-9}
\end{array}$$

G. C. D. $=x+3$, Ans.

6.

$$\begin{array}{l|l}
\multicolumn{2}{l}{4x^4-12x^3+5x^2+6x+1\;\underline{|\,2x^2-3x-1}, \text{ Ans.}} \\
\multicolumn{2}{l}{4x^4} \\
\hline
4x^2-3x & -12x^3 \\
\qquad\;\; -3x & -12x^3+9x^2 \\
\hline
4x^2-6x-1 & -4x^2+6x+1 \\
\qquad\qquad\;\; -1 & -4x^2+6x+1 \\
\hline
\end{array}$$

7.

Let $x =$ the greater number,

and $y =$ the less.

By the conditions, $x + y = a$ (1)

$x - y = b$ (2)

Adding (1) and (2), $2x = a + b$

Whence, $x = \frac{a+b}{2}$

Subtracting (2) from (1), $2y = a - b$

Whence, $y = \frac{a-b}{2}$

$$\text{Ans.} \begin{cases} \text{The greater is } \frac{a+b}{2}. \\ \text{The less is } \frac{a-b}{2}. \end{cases}$$

8.

$$\begin{aligned} 3a\sqrt[5]{b} - \sqrt[10]{a^{10}b^2} &= 3a\sqrt[5]{b} - \sqrt[10]{a^{10}}\sqrt[10]{b^2} \\ &= 3a\sqrt[5]{b} - a\sqrt[5]{b} \\ &= 2a\sqrt[5]{b}, \text{ Ans.} \end{aligned}$$

9.

Given $2x = 4 + \frac{6}{x}$

Dividing by 2 and clearing of fractions,

$$x^2 = 2x + 3$$

That is, $x^2 - 2x = 3$

Completing the square, $x^2 - 2x + 1 = 4$

Extracting the square root, $x - 1 = \pm 2$

That is, $x = 1 \pm 2$

Whence, $x = 3$, or -1, Ans.

10.

Here, $a = 24,\quad l = 192,\quad$ and $m = 2$

Hence (Art. 347), $r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}} = \left(\frac{192}{24}\right)^{\frac{1}{2+1}} = (8)^{\frac{1}{3}} = 2$

Then the means are,

24×2 and 24×2^2; that is, 48 and 96, Ans.

(413.)

1.

$$(4a^{-3}b^2x^{-4})^{-4} = 4^{-4}a^{12}b^{-8}x^{16} = \frac{a^{12}x^{16}}{4^4b^8} = \frac{a^{12}x^{16}}{256\,b^8}, \text{ Ans.}$$

2.

$$\frac{x^2 - 2x - 15}{x^2 + 10x + 21} = \frac{(x-5)(x+3)}{(x+7)(x+3)} = \frac{x-5}{x+7}, \text{ Ans.}$$

3.

$$n^3 - 2n^2 + n = n(n^2 - 2n + 1) = n(n-1)(n-1), \text{ Ans.}$$

$$x^8 - 1 = (x^4+1)(x^4-1) = (x^4+1)(x^2+1)(x^2-1)$$
$$= (x^4+1)(x^2+1)(x+1)(x-1), \text{ Ans.}$$

$$x^3 - n^3y^3 = (x - ny)(x^2 + nxy + n^2y^2), \text{ Ans.}$$

$$x^6 + y^6 = (x^2+y^2)(x^4 - x^2y^2 + y^4), \text{ Ans.}$$

4.

$$\frac{2}{\sqrt{5}-\sqrt{2}} = \frac{2(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})} = \frac{2(\sqrt{5}+\sqrt{2})}{5-2}$$
$$= \tfrac{2}{3}(\sqrt{5}+\sqrt{2}), \text{ Ans.}$$

5.

$$\{\sqrt{x} - 2 + \sqrt{-3}\}\,\{\sqrt{x} + 2 - \sqrt{-3}\}$$
$$= \{\sqrt{x} + (2 - \sqrt{-3})\}\,\{\sqrt{x} - (2 - \sqrt{-3})\}$$
$$= x - (2 - \sqrt{-3})^2$$
$$= x - (4 - 4\sqrt{-3} - 3)$$
$$= x - 1 + 4\sqrt{-3}, \text{ Ans.}$$

6.

Given $$\frac{5}{x}-\frac{3x+1}{x^2}=\frac{1}{4}$$

Clearing of fractions,
$$20x-4(3x+1)=x^2$$

Reducing, $$x^2-8x=-4$$

Completing the square, $x^2-8x+16=12$

Extracting the square root, $x-4=\pm\sqrt{12}=\pm 2\sqrt{3}$

Whence, $x=4\pm 2\sqrt{3}$, Ans.

7.

Given
$$x^2-xy=153 \qquad (1)$$
$$x+y=1 \qquad (2)$$

From (2), $$y=1-x \qquad (3)$$

Substituting in (1), $x^2-x(1-x)=153$

Reducing, $$2x^2-x=153$$

Multiplying by 4×2, and adding 1^2 to both members,
$$16x^2-8x+1=1225$$

Extracting the square root, $4x-1=\pm 35$

Transposing, $4x=1\pm 35=36$, or -34

Whence, $x=9$, or $-\frac{17}{2}$

Substituting in (3), $y=1-9$, or $1+\frac{17}{2}$

That is, $y=-8$, or $\frac{19}{2}$

$$\text{Ans.}\begin{cases}x=9, \text{ or } -\frac{17}{2}.\\ y=-8, \text{ or } \frac{19}{2}.\end{cases}$$

8.

$$\frac{1}{\sqrt{n-x^2}}=(n-x^2)^{-\frac{1}{2}}$$
$$=n^{-\frac{1}{2}}-\tfrac{1}{2}n^{-\frac{3}{2}}(-x^2)+\tfrac{3}{8}n^{-\frac{5}{2}}(-x^2)^2-\tfrac{5}{16}n^{-\frac{7}{2}}(-x^2)^3+\dots$$
$$=n^{-\frac{1}{2}}+\frac{n^{-\frac{3}{2}}x^2}{2}+\frac{3n^{-\frac{5}{2}}x^4}{8}-\frac{5n^{-\frac{7}{2}}x^6}{16}+\dots, \text{ Ans.}$$

9.

This is an infinite geometrical progression, in which $a = 1$, and $r = \frac{1}{2}$. Then, by Art. 345,

$$S = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2, \text{ Ans.}$$

10.

Let	$x =$	number B took.
Then,	$90 - x =$	" A "
Let	$y =$	sum received by each.
Then,	$\frac{y}{x} =$	price per egg received by B,
and	$\frac{y}{90 - x} =$	" " " A.

By the conditions, $$(90 - x)\frac{y}{x} = 50 \quad (1)$$

$$\frac{xy}{90 - x} = 32 \quad (2)$$

Dividing (1) by (2), $$\frac{90 - x}{x} \times \frac{90 - x}{x} = \frac{25}{16}$$

Extracting the square root, $$\frac{90 - x}{x} = \pm \frac{5}{4}$$

Taking the upper sign, $$\frac{90 - x}{x} = \frac{5}{4}$$

Clearing of fractions, $$360 - 4x = 5x$$

That is, $$9x = 360$$

Whence, $$x = 40$$

Then, $$90 - x = 50$$

Ans. $\begin{cases} \text{A took 50 eggs.} \\ \text{B " 40 "} \end{cases}$

(414.)

1.

$$(3x^2 - 2y)^5 = (3x^2)^5 - 5(3x^2)^4(2y) + 10(3x^2)^3(2y)^2$$
$$- 10(3x^2)^2(2y)^3 + 5(3x^2)(2y)^4 - (2y)^5$$
$$= 243x^{10} - 5 \,.\, 81x^8 \,.\, 2y + 10 \,.\, 27x^6 \,.\, 4y^2$$
$$- 10 \,.\, 9x^4 \,.\, 8y^3 + 5 \,.\, 3x^2 \,.\, 16y^4 - 32y^5$$
$$= 243x^{10} - 810x^8y + 1080x^6y^2 - 720x^4y^3$$
$$+ 240x^2y^4 - 32y^5, \text{ Ans.}$$

2.

Given $\qquad \frac{49}{2} - \sqrt[3]{3(x^4 + \frac{11}{81})} = \frac{47}{2}$

Transp. and uniting, $\quad - \sqrt[3]{3(x^4 + \frac{11}{81})} = -1$

Changing signs, $\quad \sqrt[3]{3(x^4 + \frac{11}{81})} = 1$

Cubing, $\quad 3(x^4 + \frac{11}{81}) = 1$

Clearing of fractions, $\quad 243x^4 + 33 = 81$

Reducing, $\quad x^4 = \frac{48}{243} = \frac{16}{81}$

Extracting the fourth root, $\quad x = \pm\frac{2}{3}$, Ans.

3.

See Art. 269.

4.

Given $\qquad \frac{x+7}{x-7} = \frac{x-7}{x+7} + 2\frac{2}{3}$

Clearing of fractions,

$$3x^2 + 42x + 147 = 3x^2 - 42x + 147 + 8x^2 - 392$$

Reducing, $\quad 8x^2 - 84x = 392$

Multiplying by 2,

$$16x^2 - 168x = 784$$

8.

Let $x =$ number of rods in the length.

Then, $\frac{5x}{8} =$ " " " breadth.

By the conditions, $\frac{5x^2}{8} = 640$

Reducing, $x^2 = 1024$

Whence, $x = 32$

And $\frac{5x}{8} = 20$

Ans. $\begin{cases} \text{Length, 32 rods.} \\ \text{Breadth, 20 rods.} \end{cases}$

9.

Let $x =$ the greater number,

and $y =$ the less.

By the conditions, $x + y = xy = x^2 - y^2$

From the first and third, $x + y = x^2 - y^2$

Dividing by $x + y$, $1 = x - y$

Whence, $y = x - 1$ (1)

Substituting in the equation, $x + y = xy$

we obtain, $x + x - 1 = x(x - 1)$

Reducing, $x^2 - 3x = -1$

Completing the square, $x^2 - 3x + \frac{9}{4} = \frac{5}{4}$

Extracting the square root, $x - \frac{3}{2} = \pm \frac{\sqrt{5}}{2}$

Whence, $x = \frac{3 \pm \sqrt{5}}{2}$

Substituting in (1), $y = \frac{3 \pm \sqrt{5}}{2} - 1$

$= \frac{1 \pm \sqrt{5}}{2}$

Ans. $\begin{cases} \text{The greater is } \frac{3 + \sqrt{5}}{2}. \\ \text{The less is } \frac{1 + \sqrt{5}}{2}. \end{cases}$

10.

Let $x =$ first figure,

and $y =$ second "

By the conditions, $x + y = 12$ (1)

$xy + 16 = 10y + x$ (2)

From (1), $y = 12 - x$ (3)

Substituting in (2), $12x - x^2 + 16 = 120 - 10x + x$

Reducing, $x^2 - 21x = -104$

Completing the square,

$$x^2 - 21x + \frac{441}{4} = \frac{441}{4} - 104 = \frac{25}{4}$$

Extracting the square root, $x - \frac{21}{2} = \pm\frac{5}{2}$

That is, $x = \frac{21}{2} \pm \frac{5}{2}$

Whence, $x = 13$, or 8

Substituting in (3), $y = 12 - 13$, or $12 - 8$

Whence, $y = -1$, or 4

Ans. The number is 84.

(415.)

1.

Given $x + a : 6a + 2 :: \frac{3}{2} : \frac{a}{3}$

By the theory of proportion, $\frac{ax}{3} + \frac{a^2}{3} = 9a + 3$

Clearing of fractions, $ax + a^2 = 27a + 9$

Transposing, $ax = -a^2 + 27a + 9$

Whence, $x = \frac{-a^2 + 27a + 9}{a}$, Ans.

2.

Given $\frac{m}{n}x = \frac{p}{9}$

Reducing, $x = \frac{p}{9} \times \frac{n}{m} = \frac{pn}{9m}$, Ans.

3.

$$\sqrt{m + 2\sqrt{mn} + n} = \sqrt{m} + \sqrt{n}, \text{ Ans.}$$

4.

When $a = 3$ and $b = 5$,

$$(\sqrt{a+b} + \sqrt{a-b})\,(\sqrt{a+b} - \sqrt{a-b})$$
$$= a + b - (a - b) = 2b = 10, \text{ Ans.}$$

5.

$$\frac{x-a}{\sqrt{x}+\sqrt{a}} = \frac{\sqrt{x}-\sqrt{a}}{3} + 2\sqrt{a}$$

Reducing, $\sqrt{x} - \sqrt{a} = \frac{\sqrt{x}-\sqrt{a}}{3} + 2\sqrt{a}$

Clearing of fractions,

$$3\sqrt{x} - 3\sqrt{a} = \sqrt{x} - \sqrt{a} + 6\sqrt{a}, \text{ Ans.}$$

6.

Let $x =$ number of days required by B.

Since A does $\frac{2}{7}$ of the work in 4 days, he would do the whole in $\frac{7}{2}$ of 4 days, or in 14 days. In one day, A performs $\frac{1}{14}$ and B performs $\frac{1}{x}$, and the two together do $\frac{5}{7}$ of the work in 6 days, or $\frac{5}{42}$ in one day. Then by the conditions,

$$\frac{1}{14} + \frac{1}{x} = \frac{5}{42}$$

Transposing and uniting, $\frac{1}{x} = \frac{2}{42} = \frac{1}{21}$

Whence, $x = 21$, Ans.

7.

$$\sqrt[3]{8^{-6}b^{3n}x^{27}} = 8^{-2}b^{n}x^{9} = \frac{b^{n}x^{9}}{64}, \text{ Ans.}$$

8.

See page 195.

9.

For the definition, see page 208. An imaginary answer denotes that the problem is arithmetically impossible. (See Art. 209.)

10.

Let $\qquad x =$ number of days.

The distance travelled by A is the sum of the terms of an arithmetical progression, whose first term is 3, and whose common difference is 2; hence, as the number of terms is x, by Art. 335,

$$\text{Distance travelled by A} = \frac{x}{2}\{3 + (3 + 2[x - 1])\}$$
$$= \frac{x}{2}\{4 + 2x\}$$
$$= x(2 + x).$$

Similarly, the distance travelled by B is the sum of the terms of an arithmetical progression, whose first term is 4, and common difference 2. Then,

$$\text{Distance travelled by B} = \frac{x}{2}\{4 + (4 + 2[x - 1])\}$$
$$= \frac{x}{2}\{6 + 2x\}$$
$$= x(3 + x)$$

By the conditions,

$$x(2 + x) + x(3 + x) = 102$$

Reducing, $\qquad 2x^2 + 5x = 102$

Multiplying by 8, and adding 25,

$$16x^2 + 40x + 25 = 816 + 25 = 841$$

Extracting the square root, $\qquad 4x + 5 = \pm 29$

That is, $\qquad 4x = -5 + 29 = 24$

Whence, $\qquad x = 6$, Ans.

(416.)

1.

Given
$$\begin{aligned} 2x &= u + y + z & (1)\\ 3y &= u + x + z & (2)\\ 4z &= u + x + y & (3)\\ u &= x - 14 & (4) \end{aligned}$$

Subtracting (3) from (2),
$$3y - 4z = z - y$$

Transposing and uniting, $4y = 5z$

Whence, $y = \frac{5z}{4}$ (5)

Substituting in (2), $\frac{15z}{4} = u + x + z$

Transposing and uniting, $\frac{11z}{4} = u + x$

Whence, $z = \frac{4}{11}(u + x)$ (6)

Therefore, from (5), $y = \frac{5}{11}(u + x)$ (7)

Substituting from (6) and (7) in (1),
$$2x = u + \frac{9}{11}(u + x) \qquad (8)$$

Substituting from (4) in (8), $2x = x - 14 + \frac{9}{11}(2x - 14)$

Clearing of fractions, $11x = -154 + 18x - 126$

Transposing and uniting, $-7x = -280$

Whence, $x = 40$

Substituting in (4), $u = 40 - 14 = 26$

Substituting in (6) and (7), $z = \frac{4}{11}(40 + 26) = 24$

$$y = \frac{5}{11}(40 + 26) = 30$$

$$\text{Ans.} \begin{cases} u = 26. \\ x = 40. \\ y = 30. \\ z = 24. \end{cases}$$

2.

Given $$x + \sqrt{a + x^2} = \frac{a^2 + a}{2\sqrt{a + x^2}}$$

Clearing of fractions,

$$2x\sqrt{a + x^2} + 2(a + x^2) = a^2 + a$$

Transposing and uniting, $$2x\sqrt{a + x^2} = a^2 - a - 2x^2$$

Squaring,

$$4x^2(a + x^2) = a^4 + a^2 + 4x^4 - 2a^3 - 4a^2x^2 + 4ax^2$$

Transp. and uniting, $$4a^2x^2 = a^4 - 2a^3 + a^2$$

Reducing, $$x^2 = \frac{a^2 - 2a + 1}{4}$$

Whence, $$x = \pm\frac{a - 1}{2}, \text{ Ans.}$$

3.

$$\frac{\sqrt{5} - \sqrt{-3}}{\sqrt{5} + \sqrt{-3}} = \frac{(\sqrt{5} - \sqrt{-3})(\sqrt{5} - \sqrt{-3})}{(\sqrt{5} + \sqrt{-3})(\sqrt{5} - \sqrt{-3})}$$

$$= \frac{5 - 2\sqrt{5}\sqrt{-3} + (-3)}{5 - (-3)}$$

$$= \frac{2 - 2\sqrt{-15}}{8} = \frac{1 - \sqrt{-15}}{4}, \text{ Ans.}$$

4.

Given $$x^2 = 10 - \frac{x}{y^{-1}} \qquad (1)$$

$$\frac{y}{x^{-1}} = 15 - y^2 \qquad (2)$$

From (1), $$x^2 + xy = 10 \qquad (3)$$

From (2), $$xy + y^2 = 15 \qquad (4)$$

Adding (3) and (4), $$x^2 + 2xy + y^2 = 25$$

Extracting the square root, $$x + y = \pm 5 \qquad (5)$$

Taking the value, $$x + y = 5 \qquad (6)$$

Substituting in (3), $$5x = 10$$

Whence, $$x = 2$$

Substituting in (6), $2+y=5$

Whence, $y=3$

Taking the value from (5), $x+y=-5$ (7)

Substituting in (3), $-5x=10$

Whence, $x=-2$

Substituting in (7), $-2+y=-5$

Whence, $y=-3$

$$\text{Ans.}\begin{cases} x=2, \text{ or } -2. \\ y=3, \text{ or } -3. \end{cases}$$

5.

Let $x =$ width of the engraving.

Then, $2x =$ its length.

Therefore, $x+6 =$ width of the Bristol board,

and $2x+6 =$ its length.

Hence, $2x^2 =$ area of the engraving,

and $(x+6)(2x+6) =$ " " Bristol board.

By the conditions,

$$(x+6)(2x+6)-2x^2=2x^2-36$$

Reducing, $2x^2+18x+36-2x^2=2x^2-36$

Therefore, $2x^2-18x=72$

Multiplying by 2, $4x^2-36x=144$

Adding 81, $4x^2-36x+81=225$

Extracting the square root, $2x-9=\pm 15$

That is, $2x=9+15=24$

Whence, $x=12$, Ans.

6.

$$\sqrt{97+56\sqrt{3}}-\sqrt{97-56\sqrt{3}}$$

$$=\sqrt{97+2\sqrt{2352}}-\sqrt{97-2\sqrt{2352}}$$

$$=\sqrt{49+2\sqrt{2352}+48}-\sqrt{49-2\sqrt{2352}+48}$$

$$=\sqrt{49}+\sqrt{48}-(\sqrt{49}-\sqrt{48})=2\sqrt{48}=8\sqrt{3}, \text{ Ans.}$$

7.

Let $x =$ number sold at 3 for 8 cents,

and $y =$ " " 5 " 12 "

Now, $\frac{28}{12} = \frac{7}{3} =$ price paid per egg.

Hence, by the conditions, $\frac{8x}{3} + \frac{12y}{5} = 624$ (1)

$\frac{7}{3}(x + y) = 560$ (2)

Reducing (1), $10x + 9y = 2340$ (3)

Reducing (2), $x + y = 240$ (4)

Multiplying (4) by 9, $9x + 9y = 2160$ (5)

Subtracting (5) from (3), $x = 180$

Substituting in (4), $180 + y = 240$

Whence, $y = 60$

Ans. { 180 were sold at 3 for 8 cents.
60 " " 5 " 12 "

8.

Let $x =$ length in rods,

and $y =$ breadth "

By the conditions, $xy = 675$ (1)

$x + y = 2(x - y)$ (2)

From (2), $3y = x$ (3)

Substituting in (1), $3y^2 = 675$

That is, $y^2 = 225$

Whence, $y = 15$

Substituting in (3), $x = 45$

Ans. { Length, 45 rods.
Breadth, 15 rods.

9.

Let $x =$ number of miles traveled by A,
and $y =$ " " " " B.

By the conditions, $x = y + 18$ (1)

Now A does $\frac{1}{9}$ of B's distance, or $\frac{y}{9}$ miles, in a day; and B does $\frac{1}{16}$ of A's distance, or $\frac{x}{16}$ miles, in a day. As, then, A does $\frac{x}{x+y}$ of the whole distance in the same number of days that B does $\frac{y}{x+y}$ of it, we have the equation,

$$\frac{\frac{x}{x+y}}{\frac{y}{9}} = \frac{\frac{y}{x+y}}{\frac{x}{16}} \qquad (2)$$

Reducing (2), $\frac{9x}{y} = \frac{16y}{x}$

Clearing of fractions, $9x^2 = 16y^2$

Extracting the square root, $3x = 4y$

Substituting from (1), $3y + 54 = 4y$

Whence, $y = 54$

Substituting in (1), $x = 72$

Ans. $\begin{cases} \text{A travels 72 miles.} \\ \text{B " 54 "} \end{cases}$

10.

See Art. 322.

(417.)

1.

$$\sqrt[3]{128} + \sqrt[3]{686} - \sqrt[3]{16} + 7\sqrt[3]{250} - 3\sqrt[3]{54}$$
$$= \sqrt[3]{64 \times 2} + \sqrt[3]{343 \times 2} - \sqrt[3]{8 \times 2} + 7\sqrt[3]{125 \times 2} - 3\sqrt[3]{27 \times 2}$$
$$= 4\sqrt[3]{2} + 7\sqrt[3]{2} - 2\sqrt[3]{2} + 35\sqrt[3]{2} - 9\sqrt[3]{2}$$
$$= 35\sqrt[3]{2}, \text{ Ans.}$$

2.

$$(\sqrt[5]{6+\sqrt[3]{8}})(\sqrt[5]{6-\sqrt[3]{8}}) = (\sqrt[5]{6+2})(\sqrt[5]{6-2})$$
$$= \sqrt[5]{8} \times \sqrt[5]{4} = \sqrt[5]{32} = 2, \text{ Ans.}$$

3.

Let	$x =$ first digit,	
	$y =$ second "	
and	$z =$ third "	
By the conditions,	$x + y + z = 15$	(1)
	$x + z = y - 1$	(2)
	$4x - z = y$	(3)
Adding (1) and (3),	$5x = 15$	
Whence,	$x = 3$	(4)
Adding (2) and (3),	$5x = 2y - 1$	
Substituting from (4),	$15 = 2y - 1$	
That is,	$2y = 16$	
Whence,	$y = 8$	
Substituting in (1),	$3 + 8 + z = 15$	
Whence,	$z = 4$	

Ans. The number is 384.

4.

Given $\quad 3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2$

Adding 3 to both members,

$$3x^2 + 15x + 3 - 2\sqrt{x^2 + 5x + 1} = 5$$

Factoring, $\quad 3(x^2 + 5x + 1) - 2\sqrt{x^2 + 5x + 1} = 5$

Multiplying by 3,

$$9(x^2 + 5x + 1) - 6\sqrt{x^2 + 5x + 1} = 15$$

Completing the square,

$$9(x^2 + 5x + 1) - 6\sqrt{x^2 + 5x + 1} + 1 = 16$$

Extracting the square root, $\quad 3\sqrt{x^2 + 5x + 1} - 1 = \pm 4$

That is, $3\sqrt{x^2+5x+1} = 1 \pm 4 = 5, \text{or} -3$ (1)

Squaring, from the first value in (1), we get,

$$9(x^2+5x+1) = 25$$

Reducing, $9x^2 + 45x = 16$

Multiplying by 4, $36x^2 + 180x = 64$

Completing the square,

$$36x^2 + 180x + 225 = 289$$

Extracting the square root,

$$6x + 15 = \pm 17$$

That is, $6x = -15 \pm 17 = 2, \text{or} -32$

Whence, $x = \frac{1}{3}, \text{or} -\frac{16}{3}$

From the second value of (1), we have,

$$\sqrt{x^2+5x+1} = -1$$

Squaring, $x^2 + 5x + 1 = 1$

Reducing, $x^2 + 5x = 0$

Factoring, $x(x+5) = 0$

Whence, $x = 0$

And $x + 5 = 0$

Therefore, $x = -5, \text{or } 0$

$$\text{Ans. } x = 0, -5, \frac{1}{3}, \text{or} -\frac{16}{3}.$$

5.

$$\frac{\sqrt{20}+\sqrt{12}+\sqrt{27}}{\sqrt{5}-\sqrt{3}} = \frac{2\sqrt{5}+2\sqrt{3}+3\sqrt{3}}{\sqrt{5}-\sqrt{3}}$$

$$= \frac{(2\sqrt{5}+5\sqrt{3})(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}$$

$$= \frac{10+5\sqrt{15}+2\sqrt{15}+15}{5-3}$$

$$= \frac{25+7\sqrt{15}}{2}, \text{ Ans.}$$

6.

$$\sqrt[3]{a^4 b^2 \sqrt[5]{a^4 b^5}} = (a^4 b^2 a^{\frac{4}{5}} b)^{\frac{1}{3}}$$
$$= (a^{\frac{24}{5}} b^3)^{\frac{1}{3}} = a^{\frac{8}{5}} b, \text{ Ans.}$$

7.

Given $\quad x^2 + xy = 5x + 4 \quad (1)$

$\qquad\quad xy + y^2 = 5y + 2 \quad (2)$

Adding (1) and (2), $\quad (x + y)^2 = 5(x + y) + 6$

Transposing, $\quad (x + y)^2 - 5(x + y) = 6$

Multiplying by 4,

$$4(x + y)^2 - 20(x + y) = 24$$

Completing the square,

$$4(x + y)^2 - 20(x + y) + 25 = 49$$

Extracting the square root,

$$2(x + y) - 5 = \pm 7$$

That is, $\quad 2(x + y) = 5 \pm 7 = 12, \text{ or } -2$

Whence, $\quad x + y = 6, \text{ or } -1 \quad (3)$

Taking the first value in (3), $\quad x = 6 - y \quad (4)$

Substituting in (2), $\quad 6y - y^2 + y^2 = 5y + 2$

Whence, $\quad y = 2$

Substituting in (4), $\quad x = 6 - 2 = 4$

Taking the second value in (3), $\quad x = -y - 1 \quad (5)$

Substituting in (2), $\quad -y^2 - y + y^2 = 5y + 2$

That is, $\quad -6y = 2$

Whence, $\quad y = -\frac{1}{3}$

Substituting in (5), $\quad x = \frac{1}{3} - 1 = -\frac{2}{3}.$

$$\text{Ans.} \begin{cases} x = 4, \text{ or } -\frac{2}{3}. \\ y = 2, \text{ or } -\frac{1}{3}. \end{cases}$$

8.

Let $x =$ A's miles per day,

and $x - 2 =$ B's " "

Then, $\frac{x}{2} =$ number of days they travel.

By the conditions, $\frac{x^2}{2} + \frac{x}{2}(x-2) = 132$

Reducing, $x^2 - x = 132$

Multiplying by 4, $4x^2 - 4x = 528$

Completing the square, $4x^2 - 4x + 1 = 529$

Extracting the square root, $2x - 1 = 23$

That is, $2x = 24$

Whence, $x = 12$

Then, $x - 2 = 10$

Ans. $\left\{\begin{array}{l}\text{A travels 12 miles a day.}\\ \text{B " 10 " "}\end{array}\right.$

9.

Let $x =$ number he bought.

By the conditions, $\frac{75}{x} = \frac{75}{x+2} + 10$

Dividing by 5, $\frac{15}{x} = \frac{15}{x+2} + 2$

Clearing of fractions, $15x + 30 = 15x + 2x^2 + 4x$

Reducing, $2x^2 + 4x = 30$

Dividing by 2, $x^2 + 2x = 15$

Completing the square, $x^2 + 2x + 1 = 16$

Extracting the square root, $x + 1 = \pm 4$

That is, $x = -1 \pm 4$

Whence, $x = 3$, Ans.

10.

The first \$25 would be at interest 24 years; the next, 23 years; and so on. Therefore the number of times that 5 per cent interest would be taken on a sum of \$25, would be the sum of the terms of the arithmetical progression,

$$24 + 23 + 22 + \ldots + 1.$$

This sum, by Art. 335, is $\frac{24}{2}(24 + 1) = 300$.

That is, the total interest is 300 times 5 per cent = 1500 per cent. And 1500 per cent of \$25 is \$375.

Also, \$25 per year, saved for 25 years, amounts to \$625.

Hence, the total amount is \$625 + \$375, which is \$1000, Ans.

EXAMINATION PAPERS.

Art. 418.

1. $$\tfrac{1}{2}(x+y)+\tfrac{1}{2}(x-y)=\frac{x}{2}+\frac{y}{2}+\frac{x}{2}-\frac{y}{2}$$
$$=\frac{2x}{2}$$
$$=x,\quad \textbf{Ans.}$$

2. $$\begin{array}{l} b\,x^2-3\,(m\,n)^{\frac{1}{2}}+1 \\ \underline{b^2\,x+(m\,n)^{\frac{1}{2}}-1} \\ b\,x^2-b^2\,x-4\,(m\,n)^{\frac{1}{2}}+2,\quad \textbf{Ans.} \end{array}$$

3. $$25^{-\frac{1}{2}}\times 25^{\frac{1}{2}}=\frac{25^{\frac{1}{2}}}{25^{\frac{1}{2}}}=1,\quad \textbf{Ans.}$$

4. $$\begin{array}{l|l|l} \underline{a^n+b^n} & a^{1+n}+a^n\,b+a\,b^n+b^{1+n} & \underline{a+b},\quad \textbf{Ans.} \\ & \underline{a^{1+n}\qquad\qquad +a\,b^n} & \\ & \qquad\quad a^n\,b+b^{1+n} & \\ & \qquad\quad \underline{a^n\,b+b^{1+n}} & \end{array}$$

5. $$36\,c^4\,d^2-16\,m^6=4\,(9\,c^4\,d^2-4\,m^6)$$
$$=4\,(3\,c^2\,d-2\,m^3)\,(3\,c^2\,d+2\,m^3),$$
Ans.

6. $$x^8-y^8=(x^4-y^4)\,(x^4+y^4)$$
$$=(x^2-y^2)\,(x^2+y^2)\,(x^4+y^4)$$
$$=(x-y)\,(x+y)\,(x^2+y^2)\,(x^4+y^4),\quad \textbf{Ans.}$$

7. $$m^9-m^5=m^5\,(m^4-1)$$
$$=m^5\,(m^2-1)\,(m^2+1)$$
$$=m^5\,(m-1)\,(m+1)\,(m^2+1),\quad \textbf{Ans.}$$

8. $$\begin{array}{r} a^2-b^2=(a-b)\,(a+b) \\ \underline{a^2-2\,a\,b+b^2=(a-b)\,(a-b)} \\ \text{G. C. D.}\ =a-b,\ \textbf{Ans.} \end{array}$$

9. $a^2-x^2=(a+x)\ (a-x)$

$4\ (a-x)=4\ (a-x)$

$\underline{a+x=(a+x)}$

L. C. M. $=4\ (a-x)\ (a+x)=4\ (a^2-x^2)$, Ans.

10. Reduce the numerator thus:—

$$\frac{m^2}{m^2-n^2}-1=\frac{m^2-m^2+n^2}{m^2-n^2}=\frac{n^2}{m^2-n^2}$$

The denominator

$$\frac{n^2}{m^2-n^2}+1=\frac{n^2+m^2-n^2}{m^2-n^2}=\frac{m^2}{m^2-n^2}$$

Then the fraction becomes

$$\frac{\dfrac{n^2}{m^2-n^2}}{\dfrac{m^2}{m^2-n^2}}=\frac{n^2}{m^2-n^2}\times\frac{m^2-n^2}{m^2}=\frac{n^2}{m^2},\quad \text{Ans.}$$

11. Given $\frac{x+2}{3}-\frac{x-3}{4}+2=x-\frac{x-1}{2}.$

L. C. D. $=12$.

Clearing of fractions,

$$4\,x+8-3\,x+9+24=12\,x-6\,x+6$$

Transposing and uniting, $-5\,x=-35$

Then, $x=\ 7$, Ans.

12. Let $x=$ smaller part,

Then, $150-x=$ larger part.

From the conditions of the problem

$$x:150-x=7:8$$

Then, $8\,x=1050-7\,x$

Transposing and uniting, $15\,x=1050$.

Whence $x=70$

And $150-x=80$

Ans. Larger part, 80. Smaller part, 70.

13. Let $x=$ number,

Then $21=\frac{m\,x}{n}$

Whence $x=\frac{21\,n}{m}$, Ans.

14. Given

$$2y + 5x = 29 \qquad (1)$$
$$2y - x = -1 \qquad (2)$$

Subtracting (2) from (1) $\quad 6x = 30$

$$x = 5$$

Substituting value of x in (2) $\quad 2y - 5 = -1$

Transposing and uniting, $\quad 2y = 4$

Whence, $\quad y = 2$

$$\text{Ans.} \begin{cases} x = 5 \\ y = 2 \end{cases}$$

15. Given

$$3x + 5y + z = 26 \qquad (1)$$
$$6x + 3y + 2z = 31 \qquad (2)$$
$$9x + 4y + 4z = 50 \qquad (3)$$

Multiply (1) by 2 $\quad 6x + 10y + 2z = 52$

And subtracting (2) $\quad 6x + 3y + 2z = 31$

We have, $\quad 7y = 21$

Therefore, $\quad y = 3$

Multiply (1) by 3, $\quad 9x + 15y + 3z = 78$

And subtracting (3), $\quad 9x + 4y + 4z = 50$

We have, $\quad 11y - z = 28$

Substituting value of y, $\quad 33 - z = 28$

Transposing and uniting, $\quad -z = -5$

Therefore, $\quad z = 5$

Substituting values of y and z in (1)

$$3x + 15 + 5 = 26$$

Transposing and uniting, $\quad 3x = 6$

Therefore, $\quad x = 2$

$$\text{Ans.} \begin{cases} x = 2 \\ y = 3 \\ z = 5 \end{cases}$$

419.

1. $a - [2b - (3c + 2b - a)] = a - [2b - 3c - 2b + a]$

$$= a - 2b + 3c + 2b - a$$
$$= 3c, \quad \text{Ans.}$$

2. $\frac{(a-d)^2 (x-y)^{-2}}{(a-b)^{-2} (x-y)} = \frac{(a-d)^2 (a-b)^2}{(x-y)^2 (x-y)} = \frac{(a-d)^2 (a-b)^2}{(x-y)^3}$, Ans.

3. $a^n b^{m-n} \div a^{n-m} d^{-n} = \frac{a^n b^{m-n}}{a^{n-m} d^{-n}}$

$$= \frac{a^{n-n+m} b^{m-n}}{d^{-n}}$$

$$= a^m b^{m-n} d^n,$$ Ans.

4. $4x^2 - 9y^2 = (2x-3y)(2x+3y)$, Ans.

5. $$\frac{\frac{c}{c-1} - 1}{1 - \frac{c}{c+1}} = \frac{\frac{c-c+1}{c-1}}{\frac{c+1-c}{c+1}} = \frac{\frac{1}{c-1}}{\frac{1}{c+1}} = \frac{c+1}{c-1},$$ Ans.

6. Given $x + 2y = 7$ (1)

$2x + 3y = 12$ (2)

Multiply (1) by 2, $2x + 4y = 14$

Subtract (2), $2x + 3y = 12$

And $y = 2$

Substitute y in (1), $x + 4 = 7$

Whence, $x = 3$

Ans. $\begin{cases} x = 3 \\ y = 2 \end{cases}$

7. $a\sqrt{48a^3 d} = a\sqrt{16a^2 \times 3ad}$

$= 4a^2\sqrt{3ad}$, Ans.

$\sqrt{\frac{45}{49}} = \sqrt{\frac{9}{49} \times 5} = \frac{3\sqrt{5}}{7}$, Ans.

8. $3\sqrt{\frac{a}{3}} \times 2\sqrt{\frac{a}{6}} = 6\sqrt{\frac{a^2}{18}} = 2a\sqrt{\frac{1}{2}} = a\sqrt{2}$, Ans.

9. Given $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}$

Clearing of fractions, $2x-10+3x=72-2x+20$

Transposing and uniting, $7x=102$

Whence, $x=14\frac{4}{7}$, Ans.

10. Let $x=$ one number

$y=$ other number

Then, $x+y=s \quad (1)$

$x-y=d \quad (2)$

Adding (1) and (2), $2x=s+d$

Whence, $x=\frac{s+d}{2}$

Subtracting (2) from (1), $2y=s-d$

Whence, $y=\frac{s-d}{2}$

$$\text{Ans.}\begin{cases}\text{One number} = \frac{s+d}{2}\\ \text{Other number} = \frac{s-d}{2}\end{cases}$$

(420.)

1. See pp. 65, 71, and 186.

2.
$$\begin{array}{l}
a^{q}+b^{st}+c^{-\frac{1}{q}}\\
\underline{a^{-2q}+c^{\frac{1}{q}}}\\
a^{-q}+a^{-2q}b^{st}+a^{-2q}c^{-\frac{1}{q}}\\
\qquad\qquad \underline{a^{q}c^{\frac{1}{q}}+b^{st}c^{\frac{1}{q}}+1}\\
a^{-q}+a^{-2q}b^{st}+a^{-2q}c^{-\frac{1}{q}}+a^{q}c^{\frac{1}{q}}+b^{st}c^{\frac{1}{q}}+1,\quad \text{Ans.}
\end{array}$$

3. $x^{\frac{1}{2}}-y^{\frac{1}{2}} \,|\, x^{\frac{3}{2}}+x^{\frac{1}{2}}y-xy^{\frac{1}{2}}-y^{\frac{3}{2}} \,|\, x+y,$ Ans.

$$x^{\frac{3}{2}} \qquad -xy^{\frac{1}{2}}$$
$$x^{\frac{1}{2}}y-y^{\frac{3}{2}}$$
$$x^{\frac{1}{2}}y-y^{\frac{3}{2}}$$

4. Given (1) $6x^4+x^3-x$

(2) $4x^3-6x^2-4x+3$

(3) $2x^3+x^2+x-1$

Suppress x in (1), multiply (2) by 3 and then divide

$$6x^3+x^2-1 \,|\, 12x^3-18x^2-12x+9 \,|\, 2$$
$$12x^3+2x^2 \qquad -2$$
$$-20x^2-12x+11$$

$$20x^2+12x-11 \,|\, 60x^3+10x^2-10 \,|\, 3x+13$$
$$60x^3+36x^2-33x$$
$$-26x^2+33x-10$$
$$260x^2-330x+100$$
$$260x^2+156x-143$$
$$-486x+243$$
$$2x-1$$

$$2x-1 \,|\, 20x^2+12x-11 \,|\, 10x+11$$
$$20x^2-10x$$
$$22x-11$$
$$22x-11$$

$(2x-1)=$ G. C. D. of (1) and (2).

$$2x-1 \,|\, 2x^3+x^2+x-1 \,|\, x^2+x+1$$
$$2x^3-x^2$$
$$2x^2+x-1$$
$$2x^2-x$$
$$2x-1$$
$$2x-1$$

G. C. D. $=2x-1$, Ans.

5. The numbers being all prime,

L. C. M. $=(x+2a)^3(x-2a)^3(x^2+4a^2)$

$=(x^2-4a^2)^3(x^2+4a^2)$, Ans.

6. Given $\frac{13a-29b}{5(a-b)}-\frac{7b-21a}{5(a-b)}+\frac{9b-11a}{5(a-b)^2}$

Reducing to common denominator,

$$\frac{(a-b)(13a-29b)}{5(a-b)^2}-\frac{(7b-21a)(a-b)}{5(a-b)^2}+\frac{9b-11a}{5(a-b)^2}$$

Multiplying out and uniting,

$$\frac{34a^2-70ab+36b^2+9b-11a}{5(a-b)^2}=\text{Ans.}$$

7. Given $\frac{1}{x}+\frac{1}{y}=a \quad (1)$

$$\frac{1}{x}+\frac{1}{z}=b \quad (2)$$

$$\frac{1}{y}+\frac{1}{z}=c \quad (3)$$

Subtracting (2) from (1), $\frac{1}{x}+\frac{1}{y}=a$

$$\frac{1}{x}+\frac{1}{z}=b$$

$$\frac{1}{y}-\frac{1}{z}=a-b \quad (4)$$

Adding (3), $\frac{1}{y}+\frac{1}{z}=c$

$$\frac{2}{y}=a-b+c$$

Whence, $y=\frac{2}{a-b+c}$

Subtracting (4) from (3), $\frac{2}{z}=c-a+b$

Whence, $z=\frac{2}{c-a+b}$

Substituting value of y in (1),

$$\frac{1}{x}+\frac{1}{\frac{2}{a-b+c}}=a$$

Reducing, $\frac{1}{x}+\frac{a-b+c}{2}=a$

Clearing of fractions, $2 + a x - b x + c x = 2 a x$

Uniting, $- a x - b x + c x = -2$

Factoring, $x (a + b - c) = 2$

Whence, $x = \frac{2}{a + b - c}$

Ans. $\begin{cases} x = \frac{2}{a + b - c} \\ y = \frac{2}{a - b + c} \\ z = \frac{2}{b + c - a} \end{cases}$

8. Given, $\sqrt{x + a} = \sqrt{x} + a$

Squaring, $x + a = x + 2 a \sqrt{x} + a^2$

Transp., uniting, and changing signs,

$$2 a \sqrt{x} = a - a^2$$

Dividing by a, $2 \sqrt{x} = 1 - a$

Squaring, $4 x = 1 - 2 a + a^2$

Whence, $x = \frac{1 - 2 a + a^2}{4}$

$$= \frac{(1 - a)^2}{4}, \text{ Ans.}$$

9. $$5 a^2 b^2 \sqrt{c} = 5 a^2 b^2 \sqrt{c}$$
$$\sqrt{a^4 b^4 c} = a^2 b^2 \sqrt{c}$$
$$\overline{4 a^2 b^2 \sqrt{c}}, \text{ Ans.}$$

10. Let $x =$ money A had at first,
$y =$ money B had at first.

Then $x - 10 = 2 (y + 10) - 35$ (1)

$y - 10 = \frac{7 (x + 10)}{19}$ (2)

Reducing, $x - 2 y = - 5$ (3)

$19 y - 7 x = 260$ (4)

Multiplying (3) by 7, $- 14 y + 7 x = - 35$

And adding to (4), $19 y - 7 x = 260$

We have, $5 y = 225$

Whence, $\underline{y = 45}$

Substituting y in (3) $x - 90 = -5$

We have $x = 85$

Ans., $\begin{cases} \text{A had 85 s.} \\ \text{B had 45 s.} \end{cases}$

11. Given, $\frac{x-1}{x-4} + 2x = 12$

Clearing of fractions, $x - 1 + 2x^2 - 8x = 12x - 48$

Transp. and uniting, $2x^2 - 19x = -47$

Multiplying by 2, $4x^2 - 38x = -94$

Completing the square, $4x^2 - 38x + \frac{361}{4} = -94 + \frac{361}{4}$

$$= \frac{-15}{4}$$

Extracting square root, $2x - \frac{19}{2} = \frac{\pm\sqrt{-15}}{2}$

Transposing, $2x = \frac{19 \pm \sqrt{-15}}{2}$

Whence, $x = \frac{19 \pm \sqrt{-15}}{4}$, Ans.

12. Given, $x^{2n} - ax^n = b$

Completing the square, $x^{2n} - ax^n + \frac{a^2}{4} = \frac{a^2}{4} + b$

$$= \frac{a^2 + 4b}{4}$$

Extracting square root, $x^n - \frac{a}{2} = \pm \frac{\sqrt{a^2 + 4b}}{2}$

Transposing, $x^n = \frac{a \pm \sqrt{a^2 + 4b}}{2}$

Whence, $x = \left(\frac{a \pm \sqrt{a^2 + 4b}}{2}\right)^{\frac{1}{n}}$, Ans.

(421.)

1. See pp. 12, 9, 10, 11, 13, 103.

2. $a^{\frac{1}{2}}$; $a^{\frac{1}{3}}$; $(a^2 + b^2 - 2ab)^{\frac{1}{2}}$ or $a - b$.

3. $\frac{1}{a^2}$; $\frac{a}{b}$; $\frac{b^2}{a^2}$.

4. $(a-b\sqrt{-1})\,(a+b\sqrt{-1})=a^2-b^2\,(\sqrt{-1})^2$

$$=a^2-b^2\,(-1)$$

$$=a^2+b^2, \text{ Ans.}$$

$$\begin{array}{l} a-b\sqrt{-1} \\ \underline{a+c\sqrt{-1}} \\ a^2-a\,b\sqrt{-1} \\ \underline{\quad+a\,c\sqrt{-1}-b\,c\,(-1)} \\ a^2-(a\,b-a\,c)\,\sqrt{-1}+b\,c, \text{ Ans.} \end{array}$$

5. $(a-b\sqrt{-1})^3=a^3-3\,a^2\,b\sqrt{-1}+3\,a\,(b\sqrt{-1})^2$

$$-(b\sqrt{-1})^3$$

$$=a^3-3\,a^2\,b\sqrt{-1}-3\,a\,b^2+b^3\sqrt{-1}, \text{ Ans.}$$

$$(a^3-2\,a^2\,b+a\,b^2)^{\frac{1}{2}}=(a(a^2-2\,a\,b+b^2))^{\frac{1}{2}}$$

$$=a^{\frac{1}{2}}\,(a-b), \text{ Ans.}$$

6. 1st. Given, $\frac{a^2-x^2}{a+x}-\frac{a^2-x^2}{a-x}=b$

Reducing to lowest terms, $a-x-(a+x)=b$

Removing paren. and uniting, $-2\,x=b$

Whence, $x=-\frac{b}{2}$.

2d. Given, $\frac{a}{x^{-1}}+b\,x^0+c=0$

Since $\frac{1}{x^{-1}}=x$; and $x^0=1$

We have, $a\,x+b+c=0$

Transposing, $a\,x=-b-c$

Whence, $x=-\frac{b+c}{a}$, Ans.

3d. Given, $\frac{x-1}{2}-\frac{x-2}{3}=\frac{x+1}{6}$

Clearing of fractions, $3\,x-3-2\,x+4=x+1$

Transp. and uniting, $0=0$

An Identical Equation, see p. 104.

4th. Given, $\frac{a^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}{a^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}=\frac{1}{a}$

Clearing of fractions, $a^{\frac{3}{2}} - a(a-x)^{\frac{1}{2}} = a^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}$

Transp. and changing signs, $a(a-x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}$

$$= a^{\frac{3}{2}} - a^{\frac{1}{2}}$$

Factoring, $(a+1)(a-x)^{\frac{1}{2}} = a^{\frac{1}{2}}(a-1)$

Squaring, $(a+1)^2(a-x) = a(a-1)^2$

Multiplying out, $a^3 + 2a^2 + a - x(a^2 + 2a + 1)$

$$= a^3 - 2a^2 + a$$

Uniting and transp., $-x(a^2 + 2a + 1) = -4a^2$

Or, $x(a+1)^2 = 4a^2$

Whence, $x = \dfrac{4a^2}{(a+1)^2}$, Ans.

(422.)

1. See pp. 53, 54.
$(3a + 2b)^2 = 9a^2 + 12ab + 4b^2$, **Ans.**

2. See p. 55.
$(a + 2b)(a - 2b) = a^2 - 4b^2$, Ans.

3.

$$\begin{array}{l|l|l}
ab + ac - bc & a^2b^2 - a^2c^2 + 2abc^2 - b^2c^2 & ab - ac + bc \\
 & a^2b^2 + a^2bc - ab^2c & \text{Ans.} \\
 & -a^2bc + ab^2c - a^2c^2 + 2abc^2 - b^2c^2 & \\
 & -a^2bc \qquad - a^2c^2 + abc^2 & \\
 & ab^2c + abc^2 - b^2c^2 & \\
 & ab^2c + abc^2 - b^2c^2 &
\end{array}$$

4.

$$\begin{array}{l|l|l}
3x^3 - 13x^2 + 23x - 21 & 6x^3 + x^2 - 44x + 21 & 2 \\
 & 6x^3 - 26x^2 + 46x - 42 & \\
 & 27x^2 - 90x + 63 & \\
\text{Or,} & 3x^2 - 10x + 7 &
\end{array}$$

$$\begin{array}{l|l|l}
3x^2 - 10x + 7 & 3x^3 - 13x^2 + 23x - 21 & x - 1 \\
 & 3x^3 - 10x^2 + 7x & \\
 & -3x^2 + 16x - 21 & \\
 & -3x^2 + 10x - 7 & \\
 & 6x - 14 &
\end{array}$$

Or, $\quad 3x-7 \,|\, 3x^2-10x+7 \,|\, x-1$

$$\begin{array}{r} 3x^2-7x \\ \hline -3x+7 \\ -3x+7 \\ \hline \end{array}$$

Ans. $= 3x-7$.

5. Given, $\quad \dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}+\dfrac{16x-x^2}{x^2-4}$

This is the same as, $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}-\dfrac{16x-x^2}{4-x^2}$

(See § 121)

Then, $\quad$ L. C. D. $=4-x^2$

Reducing to L. C. D. and writing as a single fraction, we have,

$$\frac{(3+2x)(2+x)-(2-3x)(2-x)-(16x-x^2)}{4-x^2}$$

Multiplying out,

$$\frac{6+7x+2x^2-4+8x-3x^2-16x+x^2}{4-x^2}$$

Uniting, $\quad \dfrac{2-x}{4-x^2}$

Reducing, $\quad \dfrac{1}{2+x}=$ Ans.

6. Given $\quad \dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5} \quad (1)$

$8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3} \quad (2)$

Reducing (1), $\quad 15x-25y+30=4x+2y$

$11x-27y=-30 \quad (3)$

Reducing (2), $\quad 96-3x+6y=6x+4y$

$9x-2y=96 \quad (4)$

From (3), $\quad x=\dfrac{27y-30}{11}$

From (4), $\quad x=\dfrac{2y+96}{9}$

Then $\quad \dfrac{27y-30}{11}=\dfrac{2y+96}{9}$

Clearing of fractions, $243y - 270 = 22y + 1056$

Transp. and uniting, $221y = 1326$

Whence, $y = 6$

Substituting y in (4), $9x - 12 = 96$

Transp. and uniting, $9x = 108$

Whence, $x = 12$

Ans., $\begin{cases} x = 12 \\ y = 6 \end{cases}$

7. $\sqrt{300} = \sqrt{3 \times 100} = 10\sqrt{3}$

$\sqrt{75} = \sqrt{3 \times 25} = 5\sqrt{3}$

$15\sqrt{3}$, Ans.

8. Given, $\frac{x}{7} + \frac{21}{x+5} = \frac{23}{7}$

Clearing of fractions, $x^2 + 5x + 147 = 23x + 115$

Transp. and uniting, $x^2 - 18x = -32$

Completing the square, $x^2 - 18x + 81 = 49$

Extracting square root, $x - 9 = \pm 7$

Transposing, $x = 9 \pm 7$

Whence, $x = 16$ or 2, Ans.

(423.)

1. $6a - [4b - \{4a - (6a - 4b)\}]$

$$\begin{aligned} &= 6a - [4b - \{4a - 6a + 4b\}] \\ &= 6a - [4b - 4a + 6a - 4b] \\ &= 6a - 4b + 4a - 6a + 4b \\ &= 4a, \text{ Ans.} \end{aligned}$$

2. $a^{-n} - b^{2n} \mid a^{-3n} - b^{6n} \mid a^{-2n} + a^{-n}b^{2n} + b^{4n}$, Ans.

$$\begin{array}{l} a^{-3n} - a^{-2n}b^{2n} \\ \hline \qquad a^{-2n}b^{2n} - b^{6n} \\ \qquad a^{-2n}b^{2n} - a^{-n}b^{4n} \\ \hline \qquad\qquad a^{-n}b^{4n} - b^{6n} \\ \qquad\qquad a^{-n}b^{4n} - b^{6n} \\ \hline \end{array}$$

3. 1st.
$$\frac{a^n}{a^m}=a^{n-m}$$

If $m=n$ we have

$$\frac{a^n}{a^m}=a^{n-m}$$
$$=a^{n-n}$$
$$=a^0$$

Also, $$\frac{a^n}{a^m}=\frac{a^n}{a^n}$$
$$=1$$

Therefore, $a^0=1$ See Ax. 7.

2d.
$$\frac{a^n}{a^m}=a^{n-m}$$

If $n=0$

$$\frac{a^n}{a^m}=a^{n-m}$$
$$=a^{0-m}$$
$$=a^{-m}$$

Also, $$\frac{a^n}{a^m}=\frac{a^0}{a^m}=\frac{1}{a^m}$$

Therefore $a^{-m}=\frac{1}{a^m}$. See Ax. 7.

4.
$$a^{4m}-b^{4n}=(a^{2m}-b^{2n})\ (a^{2m}+b^{2n})$$
$$=(a^m-b^n)\ (a^m+b^n)\ (a^{2m}+b^{2n}), \text{ Ans.}$$

5.
$$a^3+a^2b-ab^2-b^3=a^2(a+b)-b^2(a+b)$$
$$=(a+b)\ (a^2-b^2)$$
$$=(a+b)\ (a+b)\ (a-b)$$
$$a^4-b^4=(a+b)\ (a-b)\ (a^2+b^2)$$
$$\text{G. C. D.}\ =(a+b)\ (a-b)$$
$$=a^2-b^2, \text{ Ans.}$$

6. The numerator taken by itself reduces as follows:—
$$a-1+\frac{6}{a-6}=\frac{a^2-a-6a+6+6}{a-6}$$
$$=\frac{a^2-7a+12}{a-6}.$$

The denominator reduces thus

$$a-2+\frac{3}{a-6}=\frac{a^2-2a-6a+12+3}{a-6}$$

$$=\frac{a^2-8a+15}{a-6}$$

Then placing numerator over denominator and reducing,

$$\frac{\frac{a^2-7a+12}{a-6}}{\frac{a^2-8a+15}{a-6}}=\frac{a^2-7a+12}{a^2-8a+15} \qquad \text{(A)}$$

$$=\frac{(a-4)(a-3)}{(a-5)(a-3)}$$

$$=\frac{a-4}{a-5},\ \text{Ans.}$$

Or the G. C. D. of numerator and denominator in (A) may be found instead of factoring.

7. Given,

$$3ax-2bx-\frac{c}{3}-\frac{mx}{4}=\frac{2c}{3}+\frac{3mx}{4}-n-bx+2ax$$

Clearing of fractions, transp., and uniting,

$$ax-bx-mx=c-n$$

Factoring, $x(a-b-m)=c-n$

Whence, $x=\dfrac{c-n}{a-b-m}$, Ans.

8. Let $x=$ one part.

$a-x=$ other part.

The question is ambiguous, and we may therefore have either of the following solutions: —

$a-x=mx+n$	$a-x=m(x+n)$
$x+mx=a-n$	$a-x=mx+mn$
$x(1+m)=a-n$	$x+mx=a-mn$
$x=\dfrac{a-n}{1+m}$ one part	$x(1+m)=a-mn$
	$x=\dfrac{a-mn}{1+m}$ one part.
$a-x=a-\dfrac{a-n}{1+m}$	$a-x=a-\dfrac{a-mn}{1+m}$

$$= \frac{a + am - a + n}{1 + m}$$

$$= \frac{am + n}{1 + m} \text{ other part.}$$

$$= \frac{a + am - a + mn}{1 + m}$$

$$= \frac{am + mn}{1 + m}$$

$$= \frac{m(a + n)}{1 + m} \text{ other part.}$$

9.
$$\sqrt[5]{4c^2\sqrt{2c}} = \left(4c^2(2c)^{\frac{1}{2}}\right)^{\frac{1}{5}}$$
$$= \left((32c^5)^{\frac{1}{2}}\right)^{\frac{1}{5}}$$
$$= (32c^5)^{\frac{1}{10}}$$
$$= (2^5c^5)^{\frac{1}{10}}$$
$$= 2^{\frac{1}{2}}c^{\frac{1}{2}} = \sqrt{2c}, \quad \text{Ans.}$$

10.
$$\sqrt{a + c} = (a + c)^{\frac{1}{2}}$$
$$\sqrt[3]{a + c} = (a + c)^{\frac{1}{3}}$$
$$(a + c)^{\frac{1}{2}}(a + c)^{\frac{1}{3}} = (a + c)^{\frac{5}{6}}$$
$$= \sqrt[6]{(a + c)^5}, \quad \text{Ans.}$$

(424.)

1.
$$a^3 + b^5 + c^{-\frac{1}{3}}$$
$$a^{-3} + c^{\frac{1}{4}}$$
$$1 + a^{-3}b^5 + a^{-3}c^{-\frac{1}{3}} + a^3c^{\frac{1}{4}} + b^5c^{\frac{1}{4}} + c^{-\frac{1}{12}}, \text{ Ans.}$$

2.
$$x^{\frac{1}{2}} - y^{\frac{1}{2}} \,|\, x^{\frac{3}{2}} + x^{\frac{1}{2}}y - xy^{\frac{1}{2}} - y^{\frac{3}{2}} \,|\, x + y, \quad \text{Ans.}$$
$$x^{\frac{3}{2}} \qquad - xy^{\frac{1}{2}}$$
$$x^{\frac{1}{2}}y - y^{\frac{3}{2}}$$
$$x^{\frac{1}{2}}y - y^{\frac{3}{2}}$$

3.
$$x^2 - 7x + 10 \,|\, 4x^3 - 25x^2 + 20x + 25 \,|\, 4x + 3$$
$$4x^3 - 28x^2 + 40x$$
$$3x^2 - 20x + 25$$
$$3x^2 - 21x + 30$$
$$x - 5$$

$$
\begin{array}{r|l|l}
x-5 & x^2-7x+10 & x-2 \\
 & x^2-5x & \\
\hline
 & -2x+10 & \\
 & -2x+10 & \\
\hline
\end{array}
$$

G. C. D. $=x-5$, Ans.

4. Given, $\frac{2+x}{2-x}-\frac{1-x}{1+x}=\frac{9}{5}$

Clearing of fractions, $10+5x+10x+5x^2-10+10x+5x-5x^2=18-9x+18x-9x^2$

Transp. and uniting, $9x^2+21x=18$

Completing the square, $9x^2+21x+\frac{49}{4}=\frac{121}{4}$

Extracting the square root, $3x+\frac{7}{2}=\pm\frac{11}{2}$

Transposing, $3x=-\frac{7}{2}\pm\frac{11}{2}$

Whence, $3x=2$ or -9

Dividing by 3, $x=\frac{2}{3}$ or -3, Ans.

5. Let, $x=$ time for A to do the work alone.
$y=$ time for B to do the work alone.
$z=$ time for C to do the work alone.

Then,

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{3\frac{3}{4}}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{4\frac{2}{7}}$$

$$\frac{1}{z}+\frac{1}{x}=\frac{1}{6}$$

Simplifying,

$$\frac{1}{x}+\frac{1}{y}=\frac{4}{15} \qquad (1)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{7}{30} \qquad (2)$$

$$\frac{1}{z}+\frac{1}{x}=\frac{1}{6} \qquad (3)$$

Subtracting (2) from (1), $\frac{1}{x}-\frac{1}{z}=\frac{1}{30}$ (4)

Then (3), $\frac{1}{z}+\frac{1}{x}=\frac{1}{6}$

Adding, $\frac{2}{x}=\frac{1}{5}$

Whence, $x=10$

Subtracting, $\frac{2}{z}=\frac{2}{15}$

Whence, $z=15$

Substituting x in (1), $\frac{1}{10}+\frac{1}{y}=\frac{4}{15}$

Transp. and uniting, $\frac{1}{y}=\frac{1}{6}$

Whence, $y=6$

Ans. $\begin{cases}\text{Take A 10 days.}\\ \text{Take B 6 days.}\\ \text{Take C 15 days.}\end{cases}$

Then, $\frac{1}{10}+\frac{1}{6}+\frac{1}{15}=$ amount all would do in one day.

$$=\frac{1}{3}$$

$\therefore$ Take 3 days for them to do the whole working together. Ans. 3 days.

6. Given,

$$\frac{1}{x}+\frac{1}{y}=6 \quad (1)$$

$$\frac{1}{x}+\frac{1}{z}=12 \quad (2)$$

$$\frac{1}{y}+\frac{1}{z}=10 \quad (3)$$

From (1) and (2),

$$\frac{1}{x}+\frac{1}{y}=6$$

$$\frac{1}{x}+\frac{1}{z}=12$$

$$\frac{1}{z}-\frac{1}{y}=6$$

From (3), $\frac{1}{z}+\frac{1}{y}=10$

Adding, $\frac{2}{z}=16$

Whence, $z=\frac{1}{8}$

Subtracting, $\frac{2}{y} = 4$

Whence, $y = \frac{1}{2}$

Substituting y in (1), $\frac{1}{x} + 2 = 6$

$\frac{1}{x} = 4$

$x = \frac{1}{4}$

Ans. $\begin{cases} x = \frac{1}{4} \\ y = \frac{1}{2} \\ z = \frac{1}{8} \end{cases}$

7. Let, $x =$ one number.

$y =$ other number.

Then, $\frac{x+y}{3} = 30$ (1)

And, $\frac{x-y}{5} = 24$ (2)

Clearing of fractions, $x + y = 90$ (3)

$x - y = 120$ (4)

Adding, $2x = 210$

Whence, $x = 105$

Subtracting (4) from (3), $2y = -30$

Whence, $y = -15$

Ans. $\begin{cases} \text{One number} = 105. \\ \text{Other number} = -15. \end{cases}$

8.
$$
\begin{array}{l|l}
\multicolumn{1}{r|}{x^8 - 6x^5 + 11x^2 - 6x^{-1} + x^{-4}} & \underline{x^4 - 3x + x^{-2}} \text{ Ans.} \\
\multicolumn{1}{r|}{x^8} & \\
\hline
2x^4 - 3x \;|\; -6x^5 + 11x^2 - 6x^{-1} + x^{-4} & \\
\qquad\qquad -6x^5 + 9x^2 & \\
\hline
2x^4 - 6x + x^{-2} \;|\; \quad 2x^2 - 6x^{-1} + x^{-4} & \\
\qquad\qquad\quad 2x^2 - 6x^{-1} + x^{-4} & \\
\end{array}
$$

9. See pp. 186, 208.

10. Given, $\sqrt{x} + 7 = \sqrt{119 + x}$

Squaring, $x + 14\sqrt{x} + 49 = 119 + x$

Transp. and uniting, $14\sqrt{x} = 70$

Whence, $\sqrt{x} = 5$

Squaring, $x = 25$, Ans.

(425.)

1. Let, $x =$ number of days A worked.

Then, $x - 3 =$ number of days B worked.

Therefore, $\frac{27}{x} =$ A's pay per day.

And, $\frac{18\frac{3}{4}}{x-3} =$ B's pay per day.

Then, $\frac{18\frac{3}{4}}{x-3} = \frac{27(x-3)}{x}$

Simplifying, $\frac{75x}{4(x-3)} = \frac{27(x-3)}{x}$

Clearing of fractions, $75x^2 = 108x^2 - 648x + 972$

Transp. and uniting, $-33x^2 + 648x = 972$

Dividing by -3, $11x^2 - 216x = -324$

Completing the square, $121x^2 - 2376x + 11664 = 8100$

Taking square root, $11x - 108 = \pm 90$

Transposing, $11x = 108 \pm 90$

Or, $11x = 198$ or 18

Whence, $x = 18$ or $1\frac{7}{11}$

The second value is not admissible unless the problem be differently worded.

Then, $x - 3 = 18 - 3$
$= 15$

For A's wages per day, $\frac{27}{x} = \frac{27}{18}$
$= \$1.50$

For B's wages per day, $\frac{18.75}{x-3} = \frac{18.75}{15}$
$= \$1.25$

Ans. { A worked 18 d., at \$1.50 per day.
B worked 15 d., at \$1.25 per day.

2. Given, $\left(\frac{10\sqrt[3]{a^2}}{3\sqrt[4]{b^6}}\right)^2 : x = \sqrt{\frac{5a\sqrt[3]{a^2}}{4\sqrt[5]{a^2b^9}}} : \frac{9b^{-8}}{\sqrt{5}}$

The first term reduces as follows:

$$\left(\frac{10\sqrt[3]{a^2}}{3\sqrt[4]{b^5}}\right)^2 = \left(\frac{10\,a^{\frac{2}{3}}}{3\,b^{\frac{5}{4}}}\right)^2$$

$$= \frac{100\,a^{\frac{4}{3}}}{9\,b^{\frac{5}{2}}}$$

The third term becomes,

$$\left(\frac{5\,a\sqrt[3]{a^2}}{4\sqrt[5]{a^2 b^9}}\right)^{\frac{1}{2}} = \left(\frac{5\,a^{\frac{19}{15}}}{4\,b^9}\right)^{\frac{1}{2}}$$

$$= \frac{\sqrt{5}\,a^{\frac{19}{30}}}{2\,b^{\frac{9}{2}}}$$

The fourth term,

$$\frac{9\,b^{-3}}{\sqrt{5}} = \frac{9}{\sqrt{5}\,b^3}$$

Then restating the proportion,

$$\frac{100\,a^{\frac{4}{3}}}{9\,b^{\frac{5}{2}}} : x = \frac{\sqrt{5}\,a^{\frac{19}{30}}}{2\,b^{\frac{9}{2}}} : \frac{9}{\sqrt{5}\,b^3}$$

Which becomes,

$$\frac{\sqrt{5}\,a^{\frac{19}{30}}\,x}{2\,b^{\frac{9}{2}}} = \frac{900\,a^{\frac{4}{3}}}{9\sqrt{5}\,b^{\frac{11}{2}}}$$

Or,

$$\frac{\sqrt{5}\,a^{\frac{19}{30}}\,x}{2\,b^{\frac{9}{2}}} = \frac{100\,a^{\frac{4}{3}}}{\sqrt{5}\,b^{\frac{11}{2}}}$$

Clearing of fractions, $\quad 5\,a^{\frac{19}{30}}\,b^{\frac{11}{2}}\,x = 200\,a^{\frac{4}{3}}\,b^{\frac{9}{2}}$

Whence, $\quad x = \dfrac{200\,a^{\frac{4}{3}}\,b^{\frac{9}{2}}}{5\,a^{\frac{19}{30}}\,b^{\frac{11}{2}}}$

Reducing, $\quad x = \dfrac{40\,a^{\frac{7}{10}}}{b}$

Removing fractional exponents, $\quad x = \dfrac{40\sqrt[10]{a^7}}{b}$, Ans.

3. Given,

$$\frac{\dfrac{x^2+y^2}{x^2-y^2} - \dfrac{x^2-y^2}{x^2+y^2}}{\dfrac{x-y}{x+y} + \dfrac{x+y}{x-y}}$$

The numerator reduces thus,

$$\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}=\frac{(x^2+y^2)^2-(x^2-y^2)^2}{x^4-y^4}$$
$$=\frac{x^4+2x^2y^2+y^4-x^4+2x^2y^2-y^4}{x^4-y^4}$$
$$=\frac{4x^2y^2}{x^4-y^4}$$

The denominator reduces thus,

$$\frac{x-y}{x+y}+\frac{x+y}{x-y}=\frac{(x-y)^2+(x+y)^2}{x^2-y^2}$$
$$=\frac{x^2-2xy+y^2+x^2+2xy+y^2}{z^2-y^2}$$
$$=\frac{2(x^2+y^2)}{x^2-y^2}$$

Replace numerator over denominator and reduce thus,

$$\frac{\dfrac{4x^2y^2}{x^4-y^4}}{\dfrac{2(x^2+y^2)}{x^2-y^2}}=\frac{4x^2y^2(x^2-y^2)}{2(x^2+y^2)(x^4-y^4)}$$
$$=\frac{2x^2y^2}{(x^2+y^2)(x^2+y^2)}$$
$$=\frac{2x^2y^2}{(x^2+y^2)^2},\ \text{Ans.}$$

4. Given to write out the first five and last five terms of $(x-y)^{13}$.

By the Binomial Theorem,

$(x-y)^{13}=x^{13}-13x^{12}y+78x^{11}y^2-286x^{10}y^3+715x^9y^4\ .\ .\ .\ .\ .\ -715x^4y^9+286x^3y^{10}-78x^2y^{11}+13xy^{12}-y^{13}$, Ans.

NOTE. — *The coefficients of the last five are same as those of the first taken in reverse order, so that all calculation ceases with the finding of the fifth term.*

5. Given,

$$ax+by=l \quad (1)$$
$$cy+dz=m \quad (2)$$
$$ex+fz=n \quad (3)$$

Required x.

From (1), $y=\frac{l-ax}{b}$ (4)

From (2), $y=\frac{m-dz}{c}$ (5)

From (3), $$x=\frac{n-ex}{f} \qquad (6)$$

Substituting (6) in (5), $$y=\frac{m-\dfrac{dn-dex}{f}}{c}$$
$$=\frac{fm-dn+dex}{cf} \qquad (7)$$

Putting (4) = (7), $$\frac{l-ax}{b}=\frac{fm-dn+dex}{cf}$$

Clearing of fractions, $cfl-acfx=bfm-bdn+bdex$

Transp., changing signs, and factoring,
$$x(acf+bde)=cfl+bdn-bfm$$

Whence, $$x=\frac{cfl+bdn-bfm}{acf+bde}, \text{ Ans.}$$

6. $$2x^3-x^2+8x-4=x^2(2x-1)+4(2x-1)$$
$$=(x^2+4)(2x-1)$$
$$6x^2+7x-5=6x^2-3x+10x-5$$
$$=3x(2x-1)+5(2x-1)$$
$$=(3x+5)(2x-1)$$

Hence G. C. D. $=2x-1$
And L. C. M. $=(x^2+4)(2x-1)(3x+5)$ } Ans.

Or the G. C. D. may be found by the method of division and then the L. C. M. by § 104.

7. Given, $$\frac{x+13a+3b}{5a-3b-x}-\frac{a-2b}{x+2b}=1$$

Clearing of fractions, $(x+13a+3b)(x+2b)-(5a-3b-x)(a-2b)=(5a-3b-x)(x+2b)$

Multiplying out, $x^2+13ax+5bx+26ab+6b^2-5a^2+13ab+ax-6b^2-2bx=5ax-5bx+10ab-x^2-6b^2$

Transp. and uniting, $2x^2+9ax+8bx=5a^2-29ab-6b^2$

Or, $2x^2+(9a+8b)x=5a^2-29ab-6b^2$

Multiply by 2. $4x^2+2(9a+8b)x=10a^2-58ab-12b^2$

Completing the square,
$$4x^2+2(9a+8b)x+\frac{(9a+8b)^2}{4}=\frac{121a^2-88ab+16b^2}{4}$$

Taking the square root, $2x + \frac{9a+8b}{2} = \pm \frac{11a-4b}{2}$

Transposing, $2x = -\frac{9a+8b}{2} \pm \frac{11a-4b}{2}$

$= (a-6b)$ or $(-10a-2b)$

Hence, $x = \frac{a-6b}{2}$ or $(-5a-b)$, Ans.

(426.)

1. $a^2 = a \times a$; $a^0 = 1$; $a^{-2} = \frac{1}{a^2}$; $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

2.
$$\begin{array}{r}
x^2 + x - 6 \\
x^2 - 9x + 20 \\
\hline
x^4 + x^3 - 6x^2 \qquad\qquad\qquad \\
-9x^3 - 9x^2 + 54x \qquad\quad \\
20x^2 + 20x - 120 \\
\hline
\end{array}$$

$$-x^2 + x + 12 \,)\, x^4 - 8x^3 + 5x^2 + 74x - 120 \,(\, -x^2 + 7x - 10$$

$$\begin{array}{r}
x^4 - x^3 - 12x^2 \qquad\qquad\qquad \\
\hline
-7x^3 + 17x^2 + 74x - 120 \\
-7x^3 + 7x^2 + 84x \qquad\quad \\
\hline
10x^2 - 10x - 120 \\
10x^3 - 10x - 120 \\
\hline
\end{array}$$

$7x - x^2 - 10 =$ Ans.

3. $3m^4x - 3n^4x = 3x(m^4 - n^4)$
$= 3x(m^2 - n^2)(m^2 + n^2)$
$= 3x(m-n)(m+n)(m^2+n^2)$. Ans.

4. To prove, $a^m \,.\, a^n = a^{m+n}$.

$a^m = a.a.a \,.\,.\,.\,.\,.\, m$ times. See § 19, and note.

$a^n = a.a.a \,.\,.\,.\,.\,.\, n$ times.

Multiplying, $a^m a^n = (a.a.a \,.\,.\,.\,.\,.\, m$ times$)(a.a.a \,.\,.\,.\,.\,.\, n$ times$)$

$= a.a.a \,.\,.\,.\,.\,.\, (m+n)$ times.

$= a^{m+n}$ by converse of definition of an exponent.

To prove, $\frac{a^m}{a^n}=a^{m-n}$.

$a^m=a.a\ .\ .\ .\ .\ .\ m$ times.

$a^n=a.a\ .\ .\ .\ .\ .\ n$ times.

Dividing, $\frac{a^m}{a^n}=\frac{a.a\ .\ .\ .\ .\ .\ m \text{ times.}}{a.a\ .\ .\ .\ .\ .\ n \text{ times.}}$

$=a.a\ .\ .\ .\ .\ .\ (m-n)$ times.

$=a^{m-n}$.

To prove, $(a^m)^n=a^{mn}$.

$a^m=a.a\ .\ .\ .\ .\ .\ m$ times.

$(a^m)^n=(a.a\ .\ .\ .\ .\ .\ m \text{ times})^n$

$=(a.a\ .\ .\ .\ .\ .\ m$ times$)\ (a.a\ .\ .\ .\ .\ .$ m times$)\ .\ .\ .\ .\ .\ n$ times.

$=a.a\ .\ .\ .\ .\ .\ m\,n$ times.

$=a^{mn}$.

5. Given, $$\frac{2x-9}{27}-\frac{x-3}{4}+\frac{x}{18}=\frac{25-3x}{3}$$

L. C. D. $=108$

Clearing of fractions,

$$8x-36-27x+81+6x=900-108x$$

Transp. and uniting, $95x=855$

Whence, $x=9$, Ans.

6. Let, $x=$ number of hours it takes B to overtake A.

Then, $n+x=$ time A traveled.

And, $(n+x)\,a=$ distance A traveled.

And, $bx=$ distance B traveled.

Then, $(n+x)\,a=bx$

Reducing, $an+ax=bx$

Transposing, $ax-bx=-an$

Changing signs, $bx-ax=an$

Factoring, $x(b-a)=an$

Whence, $x=\frac{an}{b-a}$

Ans., B overtakes A in $\frac{an}{b-a}$ hours.

7. For the finding of the two numbers, see (**419**), Ex. 10. For the latter part of the problem proceed thus:

$$\frac{s+d}{2} = \text{one number.}$$

$$\frac{s-d}{2} = \text{other number.}$$

Then,

$$\frac{s+d}{2} - \frac{s}{2} = \frac{s+d-s}{2}$$
$$= \frac{d}{2}, \text{ Ans.}$$

8.

$$\begin{array}{r|l}
 & x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \;\big|\; \underline{x^2 - 2xy + y^2} \\
 & x^4 \\
\hline
2x^2 - 2xy & -4x^3y + 6x^2y^2 - 4xy^3 + y^4 \\
 & -4x^3y + 4x^2y^2 \\
\hline
2x^2 - 4xy + y^2 & 2x^2y^2 - 4xy^3 + y^4 \\
 & 2x^2y^2 - 4xy^3 + y^4 \\
\hline
\end{array}$$

Ans. $x^2 - 2xy + y^2$

9.

$$7\sqrt[n]{a^{2n} b^{2mn} c^p} = 7\left(a^{2n} b^{2mn} c^p\right)^{\frac{1}{n}}$$
$$= 7 a^{\frac{2n}{n}} b^{\frac{2mn}{n}} c^{\frac{p}{n}}$$
$$= 7 a^2 b^{2m} c^{\frac{p}{n}}$$
$$= 7 a^2 b^{2m} \sqrt[n]{c^p}, \text{ Ans.}$$

10

$$6\sqrt[4]{a^2} - 3\sqrt[6]{a^3} = 6a^{\frac{1}{2}} - 3a^{\frac{1}{2}}$$
$$= 3a^{\frac{1}{2}}$$
$$= 3\sqrt{a}, \text{ Ans.}$$

(**427.**)

1. Given to reduce,

$$(a+b+c)^2 - a(b+c-a) - b(a+c-b) - c(a+b-c)$$

Take the first quantity by itself and reduce thus,

$$(a+b+c)^2 = (a + (b+c))^2$$
$$= a^2 + 2a(b+c) + (b+c)^2$$
$$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$$

Then replacing in the original expression and multiplying out the parentheses we have,

$$a^2+2ab+2ac+b^2+2bc+c^2-ab-ac+a^2-ab$$
$$-bc+b^2-ac-bc+c^2$$

Which, on uniting similar terms, becomes,

$$2a^2+2b^2+2c^2, \text{ Ans.}$$

2.

$$\left(\frac{1-x^2}{1+y}\right)\left(\frac{1-y^2}{x+x^2}\right)\left(1+\frac{x}{1-x}\right)=\frac{(1-x^2)(1-y^2)}{(1+y)(x+x^2)(1-x)}$$
$$=\frac{(1-x)(1+x)(1-y)(1+y)}{(1+y)(1+x)x(1-x)}$$
$$=\frac{1-y}{x}, \text{ Ans.}$$

3. Given, $ax+b=\frac{x}{a}+\frac{1}{b}$

Clearing of fract., $a^2bx+ab^2=bx+a$

Transposing, $a^2bx-bx=a-ab^2$

Factoring, $x(a^2b-b)=a-ab^2$

Whence, $x=\frac{a-ab^2}{a^2b-b}$

$=\frac{a(1-b^2)}{b(a^2-1)}$, Ans.

4.

$$\begin{array}{l} a^{\frac{1}{2}}+b^{\frac{1}{2}}+a^{-\frac{1}{2}}b \\ ab^{-\frac{1}{2}}-a^{\frac{1}{2}}+b^{\frac{1}{2}} \\ \hline a^{\frac{3}{2}}b^{-\frac{1}{2}}+a+a^{\frac{1}{2}}b^{\frac{1}{2}} \\ \qquad -a-a^{\frac{1}{2}}b^{\frac{1}{2}}-b \\ \qquad\qquad a^{\frac{1}{2}}b^{\frac{1}{2}}+b+a^{-\frac{1}{2}}b^{\frac{3}{2}} \\ \hline a^{\frac{3}{2}}b^{-\frac{1}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{-\frac{1}{2}}b^{\frac{3}{2}}, \text{ Ans.} \end{array}$$

5. Given, $\frac{x}{x-1}-\frac{x-1}{x}=\frac{3}{2}$

Clearing of fract., $2x^2-2x^2+4x-2=3x^2-3x$

Transp. and uniting, $-3x^2+7x=2$

Or, $3x^2-7x=-2$

Completing square, $9x^2 - 21x + \frac{49}{4} = \frac{25}{4}$

Evolving, $3x - \frac{7}{2} = \pm \frac{5}{2}$

Transposing, $3x = \frac{7 \pm 5}{2}$

$= 6 \text{ or } 1$

Whence, $x = 2 \text{ or } \frac{1}{3}$

Ans.

6. Given, $x + 5 - \sqrt{x+5} = 6$

Or, $(x+5) - (x+5)^{\frac{1}{2}} = 6$

Completing square, $(x+5) - (x+5)^{\frac{1}{2}} + \frac{1}{4} = \frac{25}{4}$

Evolving, $(x+5)^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{5}{2}$

Transposing, $(x+5)^{\frac{1}{2}} = \frac{1 \pm 5}{2}$

$= 3 \text{ or } -2$

Squaring, $x + 5 = 9 \text{ or } 4$

Transposing and uniting, $x = 4 \text{ or } -1$

Ans.

7. Given,

$$x - y = 12 \quad (1)$$
$$x^2 + y^2 = 74 \quad (2)$$

Squaring (1), $x^2 - 2xy + y^2 = 144 \quad (3)$

Subtracting (3) from (2), $2xy = -70 \quad (4)$

Add (2) and (4), $x^2 + 2xy + y^2 = 4$

Extracting the square root, $x + y = \pm 2 \quad (5)$

Add (1) and (5), $2x = 12 \pm 2 = 14 \text{ or } 10$

Whence, $x = 7 \text{ or } 5$

Subtract (1) from (5), $2y = -12 \pm 2 = -10 \text{ or } -14$

Whence, $y = -5 \text{ or } -7$

Ans., $\begin{cases} x = 7 \text{ or } 5 \\ y = -5 \text{ or } -7 \end{cases}$

8. $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}\right)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1}\right)$

$$=\left(-\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\sqrt{-1}\right)^2$$
$$=\frac{1}{4}-\frac{3}{4}(-1)$$
$$=\frac{1}{4}+\frac{3}{4}$$
$$=1. \text{ Ans.}$$

9. Given to expand $(1-x^2)^7$ by the Binomial Theorem.

$(1-x^2)^7=1^7-7.1^6.x^2+21.1^5.(x^2)^2-35.1^4.(x^2)^3+35.1^3.(x^2)^4-21.1^2.(x^2)^5+7.1.(x^2)^6-(x^2)^7$
$=1-7x^2+21x^4-35x^6+35x^8-21x^{10}+7x^{12}-x^{14}$, Ans.

10. Performing the multiplication indicated we have,

$a^2-b^2-c^2+d^2+2ad-2bc=a^2-b^2-c^2+d^2-2ad+2bc$

Cancelling, transp., and uniting, $4ad=4bc$

Dividing by 4, $ad=bc$

Dividing by db, $\frac{a}{b}=\frac{c}{d}$

Or, $a:b=c:d$, Ans.

428.

1. $50ax^{-\frac{m}{2}}+4a^{\frac{3}{2}}-100\left[a-\left(2x+\frac{m}{2}\right)\right]$

$$=50.16.5^{-2}+4.16^{\frac{3}{2}}-100[16-(10+2)]$$
$$=\frac{50.16}{25}+4(\sqrt{16})^3-400$$
$$=32+256-400$$
$$=-112, \text{ Ans.}$$

$$\begin{array}{r}
\sqrt{x^2-y^2}-2\sqrt{a+x}+3\\
4\sqrt{x^2-y^2}-3\sqrt{a+x}-1\\
\hline
-3\sqrt{x^2-y^2}+\sqrt{a+x}+4, \text{ Ans.}
\end{array}$$

3. See pp. 37, 46.

$$\frac{5\,c\,d^{-1}\,6\,x^{-4}}{8\,a^{-2}x^{2}y^{-1}z}=\frac{30\,c\,d^{-1}\,x^{-4}}{8\,a^{-2}\,x^{2}\,y^{-1}\,z}$$

$$=\frac{15\,c\,d^{-1}\,x^{-4}}{4\,a^{-2}\,x^{2}\,y^{-1}\,z}$$

$$=\frac{15\,a^{2}\,c\,y}{4\,d\,x^{6}\,z}\ \text{(see p. 46, top), Ans.}$$

4. $9\,a^{-2}+12\,a^{-1}\,b^{2}-6\,a+4\,b^{4}-4\,a^{2}\,b^{2}+a^{4}\ \big|\ 3\,a^{-1}+2\,b^{2}-a^{2}$

$$9\,a^{-2}$$

$$6\,a^{-1}+2\,b^{2}\ \big|\ 12\,a^{-1}\,b^{2}-6\,a+4\,b^{4}-4\,a^{2}\,b^{2}+a^{4}$$

$$12\,a^{-1}\,b^{2}\qquad +4\,b^{4}$$

$$6\,a^{-1}+4\,b^{2}-a^{2}\ \big|\ -6\,a\qquad -4\,a^{2}\,b^{2}+a^{4}$$

$$-6\,a\qquad -4\,a^{2}\,b^{2}+a^{4}$$

$3\,a^{-1}+2\,b^{2}-a^{2}$. Ans.

5. Multiplying numerator and denominator by $\sqrt{a+x}-\sqrt{a-x}$, we have,

$$\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\frac{a+x-2\sqrt{a^{2}-x^{2}}+a-x}{a+x-a+x}$$

$$=\frac{2\,a-2\sqrt{a^{2}-x^{2}}}{2\,x}$$

$$=\frac{a-\sqrt{a^{2}-x^{2}}}{x}\text{, Ans.}$$

6. $$\frac{1}{x+\sqrt{x^{2}-1}}+\frac{1}{x-\sqrt{x^{2}-1}}$$

$$=\frac{x-\sqrt{x^{2}-1}+x+\sqrt{x^{2}-1}}{x^{2}-x^{2}+1}$$

$$=2\,x\text{, Ans.}$$

7. Writing in the form of a fraction and rationalizing the denominator, we have,

$$\frac{1+\sqrt{-1}}{1-\sqrt{-1}}=\frac{1+2\sqrt{-1}+(-1)}{1-(-1)}$$

$$=\frac{2\sqrt{-1}}{2}$$

$$=\sqrt{-1}\text{, Ans.}$$

8. Given, $\sqrt{x-a}=\sqrt{x}-\frac{\sqrt{a}}{2}$

Squaring, $x-a=x-\sqrt{ax}+\frac{a}{4}$

Transp. and uniting, $\sqrt{ax}=a+\frac{a}{4}$

Clearing of fractions, $4\sqrt{ax}=5a$

Squaring, $16ax=25a^2$

Whence, $x=\frac{25a}{16}$, Ans.

9. See (420), Ex. 7.

10. Given, $\begin{cases} 3x^2+xy=18 & (1) \\ 4y^2+3xy=54 & (2) \end{cases}$

Putting $y=vx$, $3x^2+vx^2=18$ (3)

And, $4v^2x^2+3vx^2=54$ (4)

From (3), $x^2=\frac{18}{3+v}$ (5)

From (4), $x^2=\frac{54}{4v^2+3v}$ (6)

Comparing (5) and (6), $\frac{18}{3+v}=\frac{54}{4v^2+3v}$

Clearing of fractions, $4v^2+3v=9+3v$

Whence, $4v^2=9$

Or, $v=\pm\frac{3}{2}$.

Substituting $v=\frac{3}{2}$ in (5), $x^2=\frac{18}{3+\frac{3}{2}}=4$

Whence, $x=\pm 2$

Therefore, $y=vx=\pm 3$

Substituting $v=-\frac{3}{2}$ in (5), $x^2=\frac{18}{3-\frac{3}{2}}=12$

Whence, $x=\pm\sqrt{12}=\pm 2\sqrt{3}$

Therefore, $y=vx=\pm 3\sqrt{3}$.

Ans., $\begin{cases} x=\pm 2 \text{ or } \pm 2\sqrt{3}. \\ y=\pm 3 \text{ or } \pm 3\sqrt{3}. \end{cases}$

429.

1. Given to simplify, $\sqrt{\left(\frac{a^2-b^2}{a^3} \div \frac{a-b}{a^2}\right) + 2\,a^{-\frac{1}{2}}\,b^{\frac{1}{2}}}$

$$\sqrt{\left(\frac{a^2-b^2}{a^3} \div \frac{a-b}{a^2}\right) + 2a^{-\frac{1}{2}}\,b^{\frac{1}{2}}} = \sqrt{\frac{a^2-b^2}{a^3} \times \frac{a^2}{a-b} + \frac{2\,b^{\frac{1}{2}}}{a^{\frac{1}{2}}}}$$

$$= \sqrt{\frac{a+b}{a} + \frac{2\,b^{\frac{1}{2}}}{a^{\frac{1}{2}}}}$$

$$= \sqrt{\frac{a+b}{a} + \frac{2\,a^{\frac{1}{2}}\,b^{\frac{1}{2}}}{a}}$$

$$= \sqrt{\frac{a+2\,a^{\frac{1}{2}}\,b^{\frac{1}{2}}+b}{a}}$$

$$= \frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}}$$

$$= \frac{a+a^{\frac{1}{2}}\,b^{\frac{1}{2}}}{a} \text{ or } \frac{a+\sqrt{a\,b}}{a}, \text{ Ans.}$$

2. Given $1 - \left(\frac{z+y}{z^{\frac{1}{3}}-y^{\frac{1}{3}}} \times \frac{\sqrt[6]{z}-\sqrt[6]{y}}{z^{\frac{1}{3}}-(z\,y)^{\frac{1}{6}}+y^{\frac{1}{3}}}\right)$ to simplify.

Take the quantity in the parentheses and reduce by itself, as follows:—

$$\frac{z+y}{z^{\frac{1}{3}}-y^{\frac{1}{3}}} \times \frac{z^{\frac{1}{6}}-y^{\frac{1}{6}}}{z^{\frac{1}{3}}-(z\,y)^{\frac{1}{6}}+y^{\frac{1}{3}}} = \frac{(z+y)\,(z^{\frac{1}{6}}-y^{\frac{1}{6}})}{(z^{\frac{1}{3}}-y^{\frac{1}{3}})\,(z^{\frac{1}{3}}-(z\,y)^{\frac{1}{6}}+y^{\frac{1}{3}})}$$

$$= \frac{(z+y)\,(z^{\frac{1}{6}}-y^{\frac{1}{6}})}{(z^{\frac{1}{6}}-y^{\frac{1}{6}})\,(z^{\frac{1}{6}}+y^{\frac{1}{6}})\,(z^{\frac{1}{3}}-(z\,y)^{\frac{1}{6}}+y^{\frac{1}{3}})}$$

$$= \frac{z+y}{(z^{\frac{1}{6}}+y^{\frac{1}{6}})\,(z^{\frac{1}{3}}-(z\,y)^{\frac{1}{6}}+y^{\frac{1}{3}})}$$

$$= \frac{z+y}{z^{\frac{1}{2}}+y^{\frac{1}{2}}}$$

Then replacing in the original we have,

$$1 - \left(\frac{z+y}{z^{\frac{1}{2}}+y^{\frac{1}{2}}}\right) = \frac{z^{\frac{1}{2}}+y^{\frac{1}{2}}-z-y}{z^{\frac{1}{2}}+y^{\frac{1}{2}}}, \text{ Ans.}$$

3.
$$
\begin{array}{r|l}
 & 16x^{\frac{8}{3}} - 8x^{\frac{7}{3}} - 15x^2 + 4x^{\frac{5}{3}} + 4x^{\frac{4}{3}} \;\big|\; \underline{4x^{\frac{4}{3}} - x - 2x^{\frac{2}{3}}} \\
 & \underline{16x^{\frac{8}{3}}\qquad\qquad\qquad\qquad\qquad} \\
8x^{\frac{4}{3}} - x & -8x^{\frac{7}{3}} - 15x^2 + 4x^{\frac{5}{3}} + 4x^{\frac{4}{3}} \\
 & \underline{-8x^{\frac{7}{3}} + x^2\qquad\qquad\qquad} \\
8x^{\frac{4}{3}} - 2x - 2x^{\frac{2}{3}} & -16x^2 + 4x^{\frac{5}{3}} + 4x^{\frac{4}{3}} \\
 & \underline{-16x^2 + 4x^{\frac{5}{3}} + 4x^{\frac{4}{3}}}
\end{array}
$$

4. Given,
$$\frac{\frac{2x}{3} - \frac{5y}{12}}{\frac{7}{4}} - \frac{\frac{3x}{2} - \frac{y}{3}}{\frac{23}{2}} = 2 \qquad (1)$$

And,
$$\frac{x-y}{x+y} = \frac{1}{5} \qquad (2)$$

Reduce the first equation thus,

Numerator of first fraction, $\frac{2x}{3} - \frac{5y}{12} = \frac{8x-5y}{12}$

Numerator of second fraction, $\frac{3x}{2} - \frac{y}{3} = \frac{9x-2y}{6}$

Then the first fraction reduces,
$$\frac{\frac{8x-5y}{12}}{\frac{7}{4}} = \frac{8x-5y}{12} \times \frac{4}{7}$$
$$= \frac{8x-5y}{21}$$

And the second fraction,
$$\frac{\frac{9x-2}{6}}{\frac{23}{2}} = \frac{9x-2y}{6} \times \frac{2}{23}$$
$$= \frac{9x-2y}{69}$$

Replacing in form of an equation,
$$\frac{8x-5y}{21} - \frac{9x-2y}{69} = 2$$

Clearing of fractions, $184x - 115y - 63x + 14y = 966$

Uniting, $121x - 101y = 966 \quad (3)$

From (2), $5x - 5y = x + y$

Transp. and uniting, $4x = 6y$

Dividing by 2, $2x = 3y$

Whence, $x = \frac{3y}{2} \quad (4)$

Substituting in (3). $\frac{363y}{2} - 101y = 966$

Clearing of fractions, $363y - 202y = 1932$

Uniting, $161y = 1932$

Whence, $y = 12$

Substituting in (4), $x = \frac{36}{2}$

$= 18$

Ans. $\begin{cases} x = 18. \\ y = 12. \end{cases}$

5. Given, $x^2 + \frac{1}{x^2} - a^2 - \frac{1}{a^2} = 0$

Transposing, $x^2 + \frac{1}{x^2} = a^2 + \frac{1}{a^2}$

Clearing of fractions, $a^2x^4 + a^2 = a^4x^2 + x^2$

Transp. and factoring, $a^2x^4 - x^2(a^4 + 1) = -a^2$

Completing the square,

$$a^2x^4 - x^2(a^4 + 1) + \frac{(a^4 + 1)^2}{4a^2} = \frac{a^8 - 2a^4 + 1}{4a^2}$$

Evolving, $ax^2 - \frac{a^4 + 1}{2a} = \pm \frac{a^4 - 1}{2a}$

Transp., $ax^2 = \frac{a^4 + 1}{2a} \pm \frac{a^4 - 1}{2a}$

Uniting, $ax^2 = a^3 \text{ or } \frac{1}{a}$

Whence, $x^2 = a^2 \text{ or } \frac{1}{a^2}$

Taking square root, $x = \pm a \text{ or } \pm \frac{1}{a}$, Ans.

6. Given, $\frac{b\,x}{y+b}+\frac{a\,y}{x+a}=\frac{a+b}{2}$ (1)

$\frac{x}{a}+\frac{y}{b}=2$ (2)

Clearing of fractions in (1), $2\,b\,x^2+2\,a\,b\,x+2\,a\,y^2+2\,a\,b\,y$
$=(a+b)\,x\,y+a\,b\,x+b^2\,x+a^2\,y+a\,by+a^2\,b+a\,b^2$

Transposing and uniting terms, $2\,b\,x^2+2\,a\,y^2-(a+b)\,x\,y$
$+a\,b\,x+a\,b\,y-b^2\,x-a^2\,y=a^2\,b+a\,b^2$ (3)

From (2), $b\,x+a\,y=2\,a\,b$

Whence, $y=\frac{2\,a\,b-b\,x}{a}$ (4)

Substituting in (4), $2\,b\,x^2+\frac{2(2\,a\,b-b\,x)}{a}-(a+b)\,x$
$\frac{2\,a\,b-b\,x}{a}+a\,b\,x+b\,(2\,a\,b-b\,x)-b^2\,x-a\,(2\,a\,b-b\,x)$
$=a^2\,b+a\,b^2$

Clearing of fractions, $2\,a\,b\,x^2+8\,a^2\,b^2-8\,a\,b^2\,x+2\,b^2\,x^2$
$-2\,a^2\,b\,x-2\,a\,b^2\,x+a\,b\,x^2+b^2\,x^2+a^2\,b\,x+2\,a^2\,b^2-$
$a\,b^2\,x-a\,b^2\,x-2\,a^3\,b+a^2\,b\,x=a^3\,b+a^2\,b^2$

Dividing by b, transposing and uniting terms,

$$(3\,a+3\,b)\,x^2-12\,a\,b\,x=3\,a^3-9\,a^2\,b$$

Dividing by $3\,a+3\,b$, $x^2-\frac{4\,a\,b}{a+b}\,x=\frac{a^2\,(a-3\,b)}{a+b}$

Completing the square, $x^2-\frac{4\,a\,b}{a+b}\,x+\frac{4\,a^2\,b^2}{(a+b)^2}=\frac{a^2\,(a-b)^2}{(a+b)^2}$

Evolving, $x-\frac{2\,a\,b}{a+b}=\pm\frac{a\,(a-b)}{a+b}$

Whence, $x=\frac{2\,a\,b}{a+b}\pm\frac{a\,(a-b)}{a+b}$

$=a$ or $\frac{3\,a\,b-a^2}{a+b}$.

Substituting x in (4) and reducing,

gives $y=b$ or $\frac{3\,a\,b-b^2}{a+b}$

$$\text{Ans.}\begin{cases} x=a \text{ or } \frac{3\,a\,b-a^2}{a+b}. \\ y=b \text{ or } \frac{3\,a\,b-b^2}{a+b}. \end{cases}$$

7. Let, $x =$ time it would take A alone.
$y =$ time it would take B alone.
$z =$ time it would take C alone.

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{6} \qquad (1)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{8} \qquad (2)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{12} \qquad (3)$$

From (1), $\frac{1}{x}+\frac{1}{y}=\frac{1}{6}$

And (2), $\frac{1}{x}+\frac{1}{z}=\frac{1}{8}$

Subtracting, $\frac{1}{y}-\frac{1}{z}=\frac{1}{24} \qquad (4)$

From (3), $\frac{1}{y}+\frac{1}{z}=\frac{2}{24} \qquad (5)$

Adding, $\frac{2}{y}=\frac{1}{8}$

Whence, $y=16$

Subtracting (4) from (5), $\frac{2}{z}=\frac{1}{24}$

Whence, $z=48$

Substituting y in (1), $\frac{1}{x}+\frac{1}{16}=\frac{1}{6}$

Reducing, $\frac{1}{x}=\frac{5}{48}$

Whence, $x=9\frac{3}{5}$

$$\text{Ans.}\begin{cases}\text{A, } 9\frac{3}{5} \text{ days.}\\ \text{B, 16 days.}\\ \text{C, 48 days.}\end{cases}$$

(430.)

1. (*a.*) Factor thus,

$$\frac{a^3+a^2x-ax^2-x^3}{a^2-x^2}=\frac{a^2(a+x)-x^2(a+x)}{a^2-x^2}$$

$$=\frac{(a^2-x^2)(a+x)}{a^2-x^2}$$

$$=a+x, \text{ Ans.}$$

(*b.*) $a^{-6}b^2 = \frac{b^2}{a^6}$; $\frac{\sqrt{b}}{\sqrt[3]{a^2}} = \frac{b^{\frac{1}{2}}}{a^{\frac{2}{3}}}$

$$\left.\begin{array}{ll}\text{Then} & \frac{b^2}{a^6} \times \frac{b^{\frac{1}{2}}}{a^{\frac{2}{3}}} = \frac{b^{\frac{5}{2}}}{a^{\frac{20}{3}}} = \frac{\sqrt{b^5}}{\sqrt[3]{a^{20}}} \\ \text{And,} & \frac{b^2}{a^6} \times \frac{a^{\frac{2}{3}}}{b^{\frac{1}{2}}} = \frac{b^{\frac{3}{2}}}{a^{\frac{16}{3}}} = \frac{\sqrt{b^3}}{\sqrt[3]{a^{16}}}\end{array}\right\} \text{Ans.}$$

2. (*a.*) Given, $\frac{7x-6}{35} - \frac{x-5}{6x-101} = \frac{x}{5}$

Transposing, $\frac{7x-6}{35} - \frac{x}{5} = \frac{x-5}{6x-101}$

Or, $\frac{7x-6}{35} - \frac{7x}{35} = \frac{x-5}{6x-101}$

Reducing, $\frac{-6}{35} = \frac{x-5}{6x-101}$

Clearing of fractions, $-36x + 606 = 35x - 175$

Transp. and uniting, $-71x = -781$

Whence, $x = 11$, Ans.

(*b.*) Given, $\frac{7x+9}{4} - (x - \frac{2x-1}{9}) = 7$

Reducing, $\frac{7x+9}{4} - \frac{7x+1}{9} = 7$

Clearing of fractions, $63x + 81 - 28x - 4 = 252$

Transp. and uniting, $35x = 175$

Whence, $x = 5$, Ans.

3. (*a.*) Given, $\frac{x^2}{2} - \frac{x}{3} + 7\frac{3}{8} = 8$

Reducing, $\frac{x^2}{2} - \frac{x}{3} = \frac{5}{8}$

Clearing of fractions, $12x^2 - 8x = 15$

Multiplying by 3, $36x^2 - 24x = 45$

Adding 4 to each member, $36x^2 - 24x + 4 = 49$

Evolving, $6x - 2 = \pm 7$

Transposing, $6x = 2 \pm 7$

$= 9$ or -5

Whence, $x = 1\frac{1}{2}$ or $-\frac{5}{6}$

(*b.*) Let,

$$x = \text{1st number.}$$
$$y = \text{2d number.}$$
$$z = \text{3d number.}$$

Then,

$$xy = 15 \qquad (1)$$
$$xz = 21 \qquad (2)$$
$$y^2 + z^2 = 74 \qquad (3)$$

From (1),

$$y = \frac{15}{x} \qquad (4)$$

From (2),

$$z = \frac{21}{x} \qquad (5)$$

Then,

$$y^2 = \frac{225}{x^2} \qquad (6)$$

And,

$$z^2 = \frac{441}{x^2} \qquad (7)$$

Substituting (6) and (7) in (3),

$$\frac{225}{x^2} + \frac{441}{x^2} = 74$$

Clearing of fractions, $225 + 441 = 74x^2$

Transp., unit., and changing signs, $74x^2 = 666$

Whence, $x^2 = 9$

Extracting square root, $x = \pm 3$

Then from (4),

$$y = \frac{15}{\pm 3} = \pm 5$$

And from (5),

$$z = \frac{21}{\pm 3} = \pm 7$$

$$\text{Ans.}\begin{cases} x = \pm 3. \\ y = \pm 5. \\ z = \pm 7. \end{cases}$$

4. Using the notation of §§ 334, 335, we have given,

$$a = 1$$
$$d = 1$$
$$n = n$$

Required S.

It is seen by inspection that $l = n$; or it may be found from formula of § 334.

Then by formula of § 335,

$$\begin{aligned} S &= \tfrac{1}{2}n\,(a + l) \\ &= \tfrac{1}{2}n\,(1 + n) \\ &= \frac{n + n^2}{2}, \text{ Ans.} \end{aligned}$$

5. Given to expand $(a^3 - 2\,b^2 c)^5$ by the Binomial Theorem.

$$(a^3 - 2\,b^2 c)^5 = (a^3)^5 - 5\,(a^3)^4\,2\,b^2 c + 10\,(a^3)^3\,(2\,b^2 c)^2$$
$$- 10\,(a^3)^2\,(2\,b^2 c)^3 + 5\,(a^3)\,(2\,b^2 c)^4 - (2\,b^2 c)^5$$
$$= a^{15} - 10\,a^{12} b^2 c + 40\,a^9 b^4 c^2 - 80\,a^6 b^6 c^3 + 80\,a^3 b^8 c^4$$
$$- 32\,b^{10} c^5, \text{ Ans.}$$

(431.)

1.
$$\begin{aligned} x^2 - 3\,x + 2 &= (x - 1)\,(x - 2) \\ (x - 1)^2 &= (x - 1)\,(x - 1) \\ (x - 2)^2 &= (x - 2)\,(x - 2) \\ x^2 - 1 &= (x - 1)\,(x + 1) \\ \text{L. C. M.} &= (x - 1)^2\,(x - 2)^2\,(x + 1), \text{ Ans.} \end{aligned}$$

$$\begin{array}{r|l|l} x^3 + 3\,x^2 + 4\,x + 12 & x^3 + 4\,x^2 + 4\,x + 3 & 1 \\ & x^3 + 3\,x^2 + 4\,x + 12 & \\ \hline & \quad x^2 - 9 & \end{array}$$

Now, $\quad x^2 - 9 = (x - 3)\,(x + 3)$

And, $\quad x^3 + 3\,x^2 + 4\,x + 12 = x^2\,(x + 3) + 4\,(x + 3)$
$$= (x^2 + 4)\,(x + 3)$$

Therefore G. C. D. $\quad = x + 3$, Ans.

Or the division may be continued till the same result is reached.

2. Divide into parts as follows: —

1st part $-[\{a^2 - (b - c)^2\} - \{a^2 - (c - b)^2\}]$

2d part $(a + b + c)\,(a + b - c)\,(a + c - b)\,(b + c - a)$

Then reduce the first part thus,

$$-[\{a^2 - (b - c)^2\} - \{a^2 - (c - b)^2\}]$$
$$= -[\{a^2 - b^2 + 2\,b\,c - c^2\} - \{a^2 - c^2 + 2\,b\,c - b^2\}]$$
$$= -[a^2 - b^2 + 2\,b\,c - c^2 - a^2 + c^2 - 2\,b\,c + b^2]$$
$$= 0$$

The result will therefore depend entirely on the reduction of the second part.

Proceed thus,

$$(a + b + c)\,(a + b - c)\,(a + c - b)\,(b + c - a)$$
$$= [(a + b) + c]\,[(a + b) - c]\,[c + (a - b)]\,[c - (a - b)]$$
$$= [(a + b)^2 - c^2]\,[c^2 - (a - b)^2]$$
$$= 2\,a^2 b^2 + 2\,a^2 c^2 + 2\,b^2 c^2 - a^4 - b^4 - c^4, \text{ Ans.}$$

3. 1st. When one term is twice the algebraic product of the square roots of the other two.

2d. Because the square of a monomial is a monomial, and the square of the polynomial always contains one more term than the exponent of the power. See p. 161.

4. Given, $\frac{x}{7}-\frac{x-5}{11}+5=x-\left(\frac{2x}{77}+1\right)$

Clearing of fractions,

$$11x-7x+35+385=77x-2x-77$$

Transp. and uniting, $-71x=-497$

Whence, $x=7$, Ans.

5. Given,

$$\frac{x}{3}+\frac{y}{4}=9 \quad (1)$$

$$\frac{x}{4}+\frac{y}{5}=7 \quad (2)$$

Clearing of fractions, $4x+3y=108 \quad (3)$

$$5x+4y=140 \quad (4)$$

Multiply (3) by 5, $20x+15y=540 \quad (5)$

And (4) by 4, $20x+16y=560 \quad (6)$

Subtracting (5) from (6), $y=20$

Substituting y in (1), $\frac{x}{3}+\frac{20}{4}=9$

Reducing, $x=12$

Ans. $\begin{cases} x=12. \\ y=20. \end{cases}$

6. 1st. $(x^{\frac{2}{3}}\times x^{\frac{4}{7}})^{\frac{14}{13}}=(x^{\frac{26}{21}})^{\frac{14}{13}}$

$$=x^{\frac{4}{3}}, \text{ Ans.}$$

2d. $a^{\frac{1}{2}}\,.\,a^{-\frac{1}{3}}\,.\,a^{-\frac{1}{4}}\,.\,a^{-\frac{1}{5}}=a^{(\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5})}$

$$=a^{-\frac{17}{60}}$$

$$=\frac{1}{a^{\frac{17}{60}}}, \text{ Ans.}$$